Relativistic Quantum Mechanics
Assignment I

Joseph Pritchard - Adelaide University

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Introduction

Where specified, relative MATLAB software is included in the Appendix attached at the back.

Question I

Our Hamiltonian has two terms, the first depending on momentum, P, the second a constant, C, thus:

$$[H, L] = [P + C, L] = [P, L] + [C, L]$$

The commutator of any operator with a constant vanishes

$$[H, L] = [P, L]$$

Expressing the operators explicitly and manipulating

$$\begin{aligned} [H, L] &= [c\alpha.p, r \times p] \\ &= -c\alpha_i[\epsilon_{jkl} r_k p_l, p_i] \\ &= -c\alpha\epsilon_{jkl}(i\delta_{ik}p_l + p_i r_k p_l - p_i r_k p_l) \\ &= \alpha_i \epsilon_{jil} p_l \end{aligned} \tag{1}$$

Thus

$$[H, L] = \alpha_i \epsilon_{jil} p_l$$

Thus to make the commutator vanish we must add a negating term

$$S = -\alpha_i \epsilon_{jil} p_l$$

Thus the combination

$$J = L + S$$

Commutes with H.

Question II

Rewriting our expression in index notation we have

$$Tr(\gamma^\mu q_\mu(\gamma^\nu p_\nu + m)\gamma^\alpha q_\alpha(\gamma^\beta p_\beta + m))$$

Noting $Tr(A+B) = Tr(A) + Tr(B)$ we have

$$Tr(\gamma^\mu q_\mu \gamma^\nu p_\nu \gamma^\alpha q_\alpha \gamma^\beta p_\beta) + Tr(\gamma^\mu q_\mu \gamma^\nu p_\nu \gamma^\alpha q_\alpha m) + Tr(\gamma^\mu q_\mu m^2 \gamma^\nu q_\nu)$$

Given the 4-vectors q and p are constants and noting $Tr(cA) = cTr(A)$ for a constant c and vector A we then have

$$q_\mu p_\nu q_\alpha p_\beta Tr(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) + q_\mu p_\nu q_\alpha m Tr(\gamma^\mu \gamma^\nu \gamma^\alpha) + q_\mu q_\nu m^2 Tr(\gamma^\mu \gamma^\nu)$$

Recall

$$[\gamma^\mu, \gamma^\nu] = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I$$

Thus the third term can be reduced using

$$\begin{aligned} Tr(\gamma^\mu \gamma^\nu) &= \frac{1}{2} Tr([\gamma^\mu, \gamma^\nu]) \\ &= \frac{1}{2} Tr(2g^{\mu\nu} I) \\ &= g^{\mu\nu} Tr(I) \\ &= 4g^{\mu\nu} \end{aligned} \tag{2}$$

Leading to

$$\begin{aligned} q_\mu q_\nu m^2 Tr(\gamma^\mu \gamma^\nu) &= q^\mu q^\nu m^2 4 g_{\mu\nu} \\ &= q_\mu q^\nu m^2 \\ &= q^2 m^2 \end{aligned} \tag{3}$$

For the second term, note the trace of a product of an odd number of gamma matrices vanishes [1], thus this term is zero.

For the first term, recall the trace of a product of 4 gamma matrices obeys

$$Tr(\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta) = 4(g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha})$$

Thus the first time may be reduced as follows

$$\begin{aligned} q_\mu p_\nu q_\alpha p_\beta Tr(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) &= 4q_\mu q_\alpha p_\nu p_\beta g_{\mu\nu} g_{\alpha\beta} \\ &- 4q_\mu q_\alpha p_\nu p_\beta g_{\mu\alpha} g_{\nu\beta} \\ &+ 4q_\mu q_\alpha p_\nu p_\beta g_{\mu\beta} g_{\nu\alpha} \\ &= 4(q.p)(q.p) \\ &- 4q^2p^2 \\ &+ 4(q.p)(q.p) \end{aligned} \tag{4}$$

Thus the combined result is

$$\begin{aligned} Tr(\gamma^\mu q_\mu(\gamma^\nu p_\nu + m)\gamma^\alpha q_\alpha(\gamma^\beta p_\beta + m)) &= q^2m^2 + 8(q.p)^2 - 4q^2p^2 \\ &= q^2(m^2 - 4p^2) + 8(q.p)^2 \end{aligned} \tag{5}$$

b) MATLAB software is produced which, given any 4x4 matrix, produces the corresponding overlap with each of the sixteen Γ matrices. This is achieved via use of the overlap equation

$$c_J = \frac{1}{4}Tr(M\Gamma_J^{-1})$$

Where C_J is the overlap of 4X4 matrix M with the Jth gamma matrix. The convention used for the gamma matrices is that defined in the course notes [1]. MATLAB software produced may be viewed in the appendix. The resulting overlaps produce the following results

$$\sigma_{01}\gamma_5 = i\sigma_{23}$$

$$\sigma_{02}\gamma_5 = -i\sigma_{13}$$

$$\sigma_{03}\gamma_5 = i\sigma_{12}$$

$$\sigma_{12}\gamma_5 = -i\sigma_{03}$$

$$\sigma_{13}\gamma_5 = i\sigma_{02}$$

$$\sigma_{23}\gamma_5 = -i\sigma_{01}$$

For the sigma-sigma case

$$\sigma_{01}\sigma_{01} = -I$$

$$\sigma_{02}\sigma_{02} = -I$$

$$\sigma_{03}\sigma_{03} = -I$$

$$\sigma_{12}\sigma_{12} = I$$

$$\sigma_{13}\sigma_{13} = I$$

$$\sigma_{23}\sigma_{23} = I$$

For the gamma-sigma-gamma case

$$\gamma^0\sigma_{01}\gamma^0 = -\sigma_{01}$$
$$\gamma^1\sigma_{01}\gamma^1 = \sigma_{01}$$
$$\gamma^2\sigma_{01}\gamma^2 = -\sigma_{01}$$
$$\gamma^3\sigma_{01}\gamma^3 = -\sigma_{01}$$

$$\gamma^0\sigma_{02}\gamma^0 = -\sigma_{02}$$
$$\gamma^1\sigma_{02}\gamma^1 = -\sigma_{02}$$
$$\gamma^2\sigma_{02}\gamma^2 = \sigma_{02}$$
$$\gamma^3\sigma_{02}\gamma^3 = -\sigma_{02}$$

$$\gamma^0\sigma_{03}\gamma^0 = -\sigma_{03}$$
$$\gamma^1\sigma_{03}\gamma^1 = -\sigma_{03}$$
$$\gamma^2\sigma_{03}\gamma^2 = -\sigma_{03}$$
$$\gamma^3\sigma_{03}\gamma^3 = \sigma_{03}$$

$$\gamma^0\sigma_{12}\gamma^0 = \sigma_{12}$$
$$\gamma^1\sigma_{12}\gamma^1 = \sigma_{12}$$
$$\gamma^2\sigma_{12}\gamma^2 = \sigma_{12}$$
$$\gamma^3\sigma_{12}\gamma^3 = -\sigma_{12}$$

$$\gamma^0\sigma_{13}\gamma^0 = \sigma_{13}$$
$$\gamma^1\sigma_{13}\gamma^1 = \sigma_{13}$$
$$\gamma^2\sigma_{13}\gamma^2 = -\sigma_{13}$$
$$\gamma^3\sigma_{13}\gamma^3 = \sigma_{13}$$

$$\gamma^0\sigma_{23}\gamma^0 = \sigma_{23}$$
$$\gamma^1\sigma_{23}\gamma^1 = -\sigma_{23}$$
$$\gamma^2\sigma_{23}\gamma^2 = \sigma_{23}$$
$$\gamma^3\sigma_{23}\gamma^3 = \sigma_{23}$$

Question III

a) Rearranging the given Lorentz condition for Λ

$$G\Lambda G\Lambda^T = I$$

Premultiplying by G and post-multiplying by $(\Lambda^T)^{-1}$

$$\Lambda G = G(\Lambda^T)^{-1}$$

Post-multiplying by G, we find the most general form of Λ

$$\Lambda = G(\Lambda^T)^{-1}G$$

b) Consider infinitesimal variations in Λ close to the Identity

$$\Lambda_{ij} = \delta_i^j + \omega_i^j + O(\omega^2)$$

Expressing the Lorentz condition in this form

$$\Lambda G\Lambda^T = (I+\omega)^T G(I+\omega)$$

Equating terms linear in ω

$$\omega^T G = -G\omega$$
$$(G\omega)^T = -G\omega$$

Thus

$$g_{\mu\nu}\omega_\nu^\alpha = \omega_m u_n u$$

And

$$\omega_m u_n u = -\omega_n u_m u$$

Define

$$\begin{aligned}\omega_\beta^\alpha &= \omega_{\mu\nu}(g^{\mu\nu}\delta_\beta^\nu) \\ &= \frac{1}{2}\omega_{\mu\nu}(g^{\mu\alpha}\delta_\beta^\nu + g^{\nu\alpha}\delta_\beta^\mu - g^{\nu\alpha}\delta_\beta^\mu + g^{\mu\alpha}\delta_\beta^\nu)\end{aligned} \tag{6}$$

Observe the first two terms form a symmetric which vanishes when contracted with the antisymmetric $\omega_{\mu\nu}$ thus

Thus

$$\omega_\beta^\alpha = \frac{1}{2}\omega_{\mu\nu}(g^{\mu\alpha}\delta_\beta^\nu - g^{\nu\alpha}\delta_\beta^\mu)$$

Now define

$$[M^{\mu\nu}]_\beta^\alpha = i(g^{\mu\alpha}\delta_\beta^\nu - g^{\nu\alpha}\delta_\beta^\mu)$$

Thus

$$\Lambda = I - \frac{i}{2}\omega_{\mu\nu}[M^{\mu\nu}]$$

Expressing as an exponential

$$\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}[M^{\mu\nu}]}$$

c) The Dirac equation in two dimensions is

$$i\hbar(\beta\frac{1}{c}\frac{\partial}{\partial t} + \beta\alpha.\nabla_x)\psi = mc\psi$$

Thus we require out gamma matrices satisfy

$$\gamma^0 = \beta, \gamma^1 = \beta\alpha$$

To satisfy the clifford algebraic properties required for the Dirac gamma matrices we must have

$$2\alpha^2 = 2 - - > \alpha^2 = I$$

$$[\alpha, \beta]_+ = 0 - - > \alpha\beta = -\beta\alpha$$

$$\beta^2 = I$$

We thus must have

$$[\gamma^i, \gamma^j]_+ = 2g^{ij}I$$

By inspection this is satisfied by

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (7)$$

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (8)$$

d) Adjoint of a spinor transforms by post-multiplication of the inverse transform thus

$$\Lambda_i^\alpha \psi^i$$

Has adjoint

$$\bar{\psi}_i (\Lambda^{-1})_\alpha^i$$

The gamma matrices defined in (c) satisy, by inspection,

$$\gamma_0 \gamma_m u^\dagger \gamma_0 = \gamma_\mu$$

So

$$\begin{aligned}\gamma_0(\gamma_\mu\gamma_\nu)^\dagger\gamma_0 &= \gamma_0\gamma_\nu^\dagger\gamma_\mu^\dagger\gamma_0 \\ &= \gamma_0\gamma_\nu^\dagger\gamma_0\gamma_0\gamma_\mu^\dagger\gamma_0 \\ &= \gamma_\nu\gamma_m u\end{aligned} \tag{9}$$

We must now seek an expression for the spinor generators producing

$$\chi' = S(\Lambda)\chi$$

Guided by Pauli's Theorem [1] and the derivation of rotational spinor generators we search for

$$S(\Lambda)^{-1}\gamma^\alpha S(\Lambda) = \Lambda^\alpha_\beta\gamma^\beta$$

Taking the linear expansion

$$\Lambda = I - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + O(\omega^2)$$

Noting $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and

$$S(\Lambda) = I - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu} + O(\omega^2)$$

We must now find $\sigma^{\mu\nu}$. By subsitution into the assumed relation we have

$$(I + \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu})\gamma^\alpha(I - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}) = (I - \frac{i}{4}\omega_{\mu\nu}M^{\mu\nu})^\alpha_\beta\gamma^\beta + O(\omega^2)$$

Comparing terms linear in ω

$$\frac{i}{4}[\omega_{\mu\nu}\sigma^{\mu\nu}, \gamma^\alpha] = \frac{-i}{4}\omega_{\mu\nu}(M^{\mu\nu})^\alpha_\beta\gamma^\beta$$

And

$$\begin{aligned}\frac{i}{2}[\sigma^{\mu\nu}, \gamma^\alpha] &= -(M^{\mu\nu})^\alpha_\beta\gamma^\beta \\ &= g^{\mu\alpha}\gamma^\nu - g^{\nu\alpha}\gamma^\beta\end{aligned} \tag{10}$$

This condition is satisfied by

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

Thus

$$S(\Lambda) = exp(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu})$$

Now to transform our spinor observe

$$
\begin{aligned}
\gamma_0 S(\Lambda)^\dagger \gamma_0 &= \gamma_0 e^{(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu\dagger})} \\
&= e^{(\frac{i}{4}\omega_{\mu\nu}\gamma_0\sigma^{\mu\nu\dagger}\gamma_0)} \\
&= e^{(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu})} \\
&= S(\Lambda)^{-1}
\end{aligned} \tag{11}
$$

Taking motivation from the 4 dimensional case assume

$$\bar{\psi} = \psi\gamma_0$$

So

$$
\begin{aligned}
\bar{\psi}' &= (\psi')^\dagger\gamma_0 \\
&= (S(\Lambda)\psi)^\dagger\gamma_0 \\
&= \psi^\dagger S(\Lambda)^\dagger\gamma_0 \\
&= \psi^\dagger\gamma_0\gamma_0 S(\Lambda)^\dagger\gamma_0
\end{aligned} \tag{12}
$$

Since $\gamma_0 S(\Lambda)^\dagger\gamma_0 = S(\Lambda)^{-1}$ we have

$$
\begin{aligned}
\bar{\psi}' &= \psi^\dagger\gamma_0 S(\Lambda)^{-1} \\
&= \bar{\psi} S(\Lambda)^{-1}
\end{aligned} \tag{13}
$$

So $\bar{\psi} = \psi\gamma_0$ satisfies post multiplication by the inverse under transformation, as required for the adjoint.

e) From part (b) we already have 2 gamma matrices, we need 2 more to span 2x2 spinor space. Given our two matrices from (b) are

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{14}$$

$$\gamma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{15}$$

We want the remaining two matrices to allow for the formation of all possible 2x2 matrices. By inspection we find a suitable pair are

$$\gamma^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{16}$$

$$\gamma^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{17}$$

Question IV

a) Consider entries Λ_{0i} of matrix Λ and manipulate as follows

$$\begin{aligned}\frac{p_i}{m} &= \hat{p_i}\frac{|p|}{m} \\ &= \hat{p_i}\frac{|p|}{m}\frac{2(|p|+p^0)}{2(|p|+p^0)} \\ &= \hat{p_i}\frac{2|p|p^0+2|p|^2}{2m(p^0+|p|)}\end{aligned} \tag{18}$$

Now add zero

$$\hat{p_i}\frac{2|p|p^0+2|p|^2}{2m(p^0+|p|)} = \hat{p_i}\frac{2|p|p^0+2|p|^2+m^2-m^2}{2m(p^0+|p|)} \tag{19}$$

Recognise the relativistic energy expression ${p^0}^2 = |p|^2+m^2$ thus

$$\hat{p_i}\frac{1}{2}\frac{2|p|p^0+2|p|^2+m^2-m^2}{2m(p^0+|p|)} = \hat{p_i}\frac{{p^0}^2+|p|^2+2|p|p^0-m^2}{m(p^0+|p|)}$$

Splitting into fractions we have

$$\hat{p_i}\frac{1}{2}\left(\frac{p^0+|p|}{m}-\frac{m}{p^0+|p|}\right)$$

Now observe the following may be derived from $\frac{m}{p^0+|p|}$

$$\begin{aligned}\frac{m}{p^0+|p|} &= e^{-ln\left(\frac{p^0+|p|}{m}\right)} \\ &= e^{-\sqrt{ln\left(\frac{p^0+|p|}{m}\right)^2}} \\ &= e^{-\sqrt{ln\left(\frac{p^0+|p|}{m}\right)^2(\hat{p_1}^2+\hat{p_2}^2+\hat{p_3}^2)}} \\ &= e^{-\sqrt{\hat{p_1}^2 ln\left(\frac{p^0+|p|}{m}\right)^2+\hat{p_2}^2 ln\left(\frac{p^0+|p|}{m}\right)^2+\hat{p_3}^2 ln\left(\frac{p^0+|p|}{m}\right)^2}}\end{aligned} \tag{20}$$

Where use has been made of the unit nature of $\hat{p_1}^2+\hat{p_2}^2+\hat{p_3}^2$. Note also the above working shows

$$ln\left(\frac{p^0+|p|}{m}\right) = \sqrt{\hat{p_1}^2 ln\left(\frac{p^0+|p|}{m}\right)^2+\hat{p_2}^2 ln\left(\frac{p^0+|p|}{m}\right)^2+\hat{p_3}^2 ln\left(\frac{p^0+|p|}{m}\right)^2}$$

Returning to our prior expression for $\frac{p_i}{m}$ we now multiply by $ln\left(\frac{p^0+|p|}{m}\right)$

$$\frac{p_i}{m} = \frac{-\hat{p}_i ln\left(\frac{p^0+|p|}{m}\right)\frac{m}{p^0+|p|} + \hat{p}_i ln\left(\frac{p^0+|p|}{m}\right)\frac{p^0+|p|}{m}}{2ln\left(\frac{p^0+|p|}{m}\right)}$$

Define a 3-vector quantity α such that $\alpha_i = \hat{p}_i ln\left(\frac{p^0+|p|}{m}\right)$. Combining with our above derived result for $\frac{m}{p^0+|p|}$ we then may simplify our expression to

$$\frac{-\hat{\alpha}_i e^{-|\alpha|} + \hat{\alpha}_i e^{|\alpha|}}{2|\alpha|}$$

Now computing the matrix exponential of

$$\Lambda = \begin{bmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{01} & 0 & 0 & 0 \\ \omega_{02} & 0 & 0 & 0 \\ \omega_{03} & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

and comparing the resultant matrix entries to our above expression, we note the consistent mapping

$$\omega_{0i} = \alpha_i$$

and thus

$$\omega_{0i} = \hat{p}_i ln\left(\frac{p^0+|p|}{m}\right)$$

b)

Question V

By the definition of the adjoint spinor

$$\bar{\psi} = \psi^\dagger \gamma_0$$

And noting charge conjugation of the adjoint spinor is equivalent to the transpose of the spinor, we may express the charge conjugate expression as

$$\begin{aligned} \bar{\psi}_c \Gamma_J \psi_c &= \psi_c^\dagger \gamma_0 \Gamma_J \psi_c \\ &= \psi^T \gamma_0 \Gamma_J \psi_c \end{aligned} \tag{22}$$

Similarly for the non-conjugated case

$$\begin{aligned} \bar{\psi} \Gamma_J \psi &= \psi^\dagger \gamma_0 \Gamma_J \psi \\ &= \psi_c^T \gamma_0 \Gamma_J \psi \end{aligned} \tag{23}$$

By noting the dimensionality of the quantities in our expressions, we conclude the result is a constant. Namely the dimension of the vectors

$$[1 \times 4][4 \times 4][4 \times 4][4 \times 1] = [1 \times 1]$$

As such our expressions are transpose invariant thus

$$\begin{aligned} \psi_c^T \gamma_0 \Gamma_J \psi &= (\psi_c^T \gamma_0 \Gamma_J \psi)^T \\ &= \psi^T \Gamma_J^T \gamma_0^T \psi_c \\ &= \psi^T \Gamma_J^T \gamma_0 \psi_c \end{aligned} \tag{24}$$

To deduce the constants of proportionality set our two expressions equal via some constant C_J

$$\psi^T \gamma_0 \Gamma_J \psi_c = C_J \psi^T \Gamma_J^T \gamma_0 \psi_c$$

Thus

$$\psi^T \gamma_0 \Gamma_J \psi_c = \psi^T C_J \Gamma_J^T \gamma_0 \psi_c$$

So the condition is satisfied for

$$\gamma_0 \Gamma_J = C_J \Gamma_J^T \gamma_0$$

Post multiply both sides by γ_0

$$\gamma_0 \Gamma_J \gamma_0 = C_J \Gamma_J^T$$

Identify the adjoint of the Γ_J matrices

$$\begin{aligned} \Gamma_J^\dagger &= C_J \Gamma_J^T \\ \Gamma_J^{\star T} &= C_J \Gamma_J^T \\ \Gamma_J^\star &= C_J \Gamma_J \end{aligned}$$

So the constants of proportionality C_J are given by the relative signs between Γ_J and $\Gamma_J^\star$. Thus we may evaluate C_J by computing the overlap between Γ_J and $\Gamma_J^\star$ using the same MATLAB software as constructed in question 2. The results are as follows

$$C_1 = 1, C_2 = -1, C_3 = -1, C_4 = -1$$

$$C_5 = -1, C_6 = -1, C_7 = -1, C_8 = -1$$

$$C_9 = -1, C_{10} = -1, C_{11} = -1, C_{12} = 1$$

$$C_{13} = 1, C_{14} = 1, C_{15} = 1, C_{16} = 1$$

See the appendix for the script used in evaluation.

Question VI

a) Take the conjugate of

$$c\alpha_i p_i - icmw\alpha_i r_i \beta + mc^2\beta$$

Giving

$$c(\alpha_i p_i)^\dagger + icmw(\alpha_i r_i \beta)^\dagger + mc^2\beta$$
$$c(\alpha_i p_i)^\dagger + icmwr_i \beta^\dagger \alpha_i^\dagger + mc^2\beta$$

Note the hermiticity of the α and β matrices and the defining relation $\alpha\beta = -\beta\alpha$ thus

$$c\alpha_i p_i + icmwr_i \beta\alpha_i + mc^2\beta$$
$$c\alpha_i p_i - icmwr_i \alpha_i \beta + mc^2\beta$$

Which is our orignal expression and thus

$$H_D^\dagger = H_D$$

Note in the above the fact that p and r are observable operators and thus hermitian is used.

b) To solve the eigenvalue problem, consider squaring the Hamiltonian

$$H^2 = [c\alpha_i p_i - imwcr_i\alpha_i\beta + mc^2\beta]^2 \tag{25}$$

Note that the cross terms will vanish due to the anti symmetric properties of $\alpha_i\alpha_j$ and $\alpha_i\beta$ thus

$$H^2 = c^2[p_i^2 - m^2w^2r_i^2m^2c^2]I \tag{26}$$

Where I denotes the identity, which we shall assume is present onwards from here. Note the isotropic harmonic oscillator hamiltonian given by

$$H_{osc} = \frac{1}{2m}p_i^2 + \frac{1}{2}mw^2r^2$$

Manipulating our expression for H^2 we have

$$\begin{aligned} H^2 &= c^2[p_i^2 - m^2w^2r_i^2m^2c^2] \\ \frac{1}{2mc^2}H^2 &= \frac{1}{2m}p_i^2 - \frac{1}{2}mw^2r_i^2 + mc^2 \\ &= H_{osc} + \frac{1}{2}mc^2 \\ &= H_{osc} + H_{L.S} \end{aligned} \tag{27}$$

And so we see our Hamiltonian is equivalent to the hamiltonian for an isotropic harmonic oscillator plus a constant $L.S$ coupling. To find the associated energy eigenvalues observe

$$\begin{aligned} H^2\psi &= \left[H_{osc} + \frac{1}{2}mc^2\right]\psi \\ &= \left[\left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega + \frac{1}{2}mc^2\right]\psi \end{aligned} \tag{28}$$

And so

$$E_n^2 = \hbar n_T \omega + \frac{3}{2}\hbar\omega + \frac{1}{2}mc^2$$

Where $n_T = n_x + n_y + n_z$. To find the lowest energy eigenvalue set $n_T = 0$ thus

$$E_0^2 = \frac{3}{2}\hbar\omega + \frac{1}{2}mc^2 \tag{29}$$

$$\begin{aligned} E_0 &= \sqrt{\frac{3}{2}\hbar\omega + \frac{1}{2}mc^2} \\ &= \sqrt{\frac{\hbar\omega + mc^2}{2}} \end{aligned} \tag{30}$$

c) By instead represeting the above derived Energy eigenvalues in terms of spherical cooridnates, we may express our eigenvalues in terms of the angular momentum paramter j [2]. This then results in

$$E_n^2 = \left(l + 2n + \frac{3}{2}\right)\hbar\omega + mc^2$$

$$E_n = \sqrt{\left(l + 2n + \frac{3}{2}\right)\hbar\omega + mc^2}$$

Where n is the nth energy eigenstate and l is the angular momentum quantum number.

References

[1] Kamleh, W 2020, *Relativistic Quantum Mechanics Notes*, RQM, University of Adelaide, Adelaide

[2] Wolfram Alpha, *Three Dimensional Isotropic Harmonic Oscillator* http://demonstrations.wolfram.com/ThreeDimensionalIsotropicHarmonicOscillator/

Appendix

Following is the MATLAB software used in the calculation of numerous quantities.

Question II Overlaps

```
%Define Gamma matrices
G = Gamma_Generator;

disp(" ")
disp("DELTA GAMMA_5 PRODUCTS")
disp(" ")

%The 6 delta_uv * gamma_5 overlaps, G{6 --> 1}
for i = 6:1:11
    %Produce i'th delta
    delta = G{i};

    %Produce Gamma_5
    gamma_5 = -j .* G{16};

    %Product
    Target = delta * gamma_5;

    %Display formatted output
    disp("Overlap for " + num2str(i - 5) + "th delta gamma_5 product
 is:")
    V = Gamma_Overlap(Target);
    fprintf('%i%+ii ', [real(V(:)), imag(V(:))].');
    disp(' ')
end


disp(" ")
disp("DELTA DELTA PRODUCTS")
disp(" ")

%The 6 delta_uv * delta_uv overlaps, G{6 --> 1}
for i = 6:1:11
    %Produce i'th delta
    delta = G{i};

    %Product
    Target = delta * delta;

    %Display formatted output
    disp("Overlap for " + num2str(i - 5) + "th delta delta product
 is:")
    V = Gamma_Overlap(Target);
    fprintf('%i%+ii ', [real(V(:)), imag(V(:))].');
    disp(' ')
end


disp(" ")
disp("GAMMA DELTA GAMMA PRODUCTS")
```

```
disp(" ")

%The 6 delta_uv * delta_uv overlaps, G{6 --> 1}
for i = 6:1:11

    %The 4 gamma matrices, G{2 --> 5}
    for j = 2:1:5

        %Produce i'th delta
        delta = G{i};

        %Produce j'th gamma
        gamma = G{j};

        %Product
        Target = gamma * delta * gamma;

        %Display formatted output
        disp("Overlap for " + num2str(i - 5) + "th delta and " +
 num2str(j - 2) + "th gamma")
        V = Gamma_Overlap(Target);
        fprintf('%i%+ii ', [real(V(:)), imag(V(:))].');
        disp(' ')
    end
end
```

```
DELTA GAMMA_5 PRODUCTS

Overlap for 1th delta gamma_5 product is:
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 5+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 2th delta gamma_5 product is:
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -5+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 3th delta gamma_5 product is:
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 5+0i 0+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 4th delta gamma_5 product is:
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -5+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 5th delta gamma_5 product is:
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 5+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 6th delta gamma_5 product is:
0+0i 0+0i 0+0i 0+0i 0+0i -5+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i

DELTA DELTA PRODUCTS

Overlap for 1th delta delta product is:
-1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 2th delta delta product is:
```

-1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 3th delta delta product is:
-1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 4th delta delta product is:
1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 5th delta delta product is:
1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 6th delta delta product is:
1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i

GAMMA DELTA GAMMA PRODUCTS

Overlap for 1th delta and 0th gamma
0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 1th delta and 1th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 1th delta and 2th gamma
0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 1th delta and 3th gamma
0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 2th delta and 0th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 2th delta and 1th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 2th delta and 2th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 2th delta and 3th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 3th delta and 0th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 3th delta and 1th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 3th delta and 2th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 3th delta and 3th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i
0+0i 0+0i
Overlap for 4th delta and 0th gamma

```
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 4th delta and 1th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 4th delta and 2th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 4th delta and 3th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 5th delta and 0th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 5th delta and 1th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 5th delta and 2th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 5th delta and 3th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 6th delta and 0th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 6th delta and 1th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i -1+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 6th delta and 2th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
Overlap for 6th delta and 3th gamma
0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 0+0i 1+0i 0+0i 0+0i 0+0i
 0+0i 0+0i
```

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Question IV Matrix Exponential Evaluation

```
syms w1 w2 w3;

 M = [  0, w1, w2, w3;
       w1,  0,  0,  0;
       w2,  0,  0,  0;
       w3,  0,  0,  0;
     ]

 expm(M)
```

```
M =

[  0, w1, w2, w3]
[ w1,  0,  0,  0]
[ w2,  0,  0,  0]
[ w3,  0,  0,  0]


ans =

[                                    exp(-(w1^2 + w2^2 +
 w3^2)^(1/2))/2 + exp((w1^2 + w2^2 + w3^2)^(1/2))/2,
      (w1*exp((w1^2 + w2^2 + w3^2)^(1/2)) - w1*exp(-(w1^2 + w2^2
 + w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)^(1/2)),
    (w2*exp((w1^2 + w2^2 + w3^2)^(1/2)) - w2*exp(-(w1^2 + w2^2
 + w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)^(1/2)),
   (w3*exp((w1^2 + w2^2 + w3^2)^(1/2)) - w3*exp(-(w1^2 + w2^2 +
 w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)^(1/2))]
[ (w1*exp((w1^2 + w2^2 + w3^2)^(1/2)) - w1*exp(-(w1^2 + w2^2 +
 w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)^(1/2)), (w1^2*exp(-(w1^2
 + w2^2 + w3^2)^(1/2)) + 2*w2^2 + 2*w3^2 + w1^2*exp((w1^2 + w2^2
 + w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)),       (w1*w2*exp(-
(w1^2 + w2^2 + w3^2)^(1/2)) - 2*w1*w2 + w1*w2*exp((w1^2 + w2^2
 + w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)),       (w1*w3*exp(-
(w1^2 + w2^2 + w3^2)^(1/2)) - 2*w1*w3 + w1*w3*exp((w1^2 + w2^2 +
 w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2))]
[ (w2*exp((w1^2 + w2^2 + w3^2)^(1/2)) - w2*exp(-(w1^2 + w2^2 +
 w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)^(1/2)),       (w1*w2*exp(-
(w1^2 + w2^2 + w3^2)^(1/2)) - 2*w1*w2 + w1*w2*exp((w1^2 + w2^2
 + w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)), (w2^2*exp(-(w1^2 +
 w2^2 + w3^2)^(1/2)) + 2*w1^2 + 2*w3^2 + w2^2*exp((w1^2 + w2^2
 + w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)),       (w2*w3*exp(-
(w1^2 + w2^2 + w3^2)^(1/2)) - 2*w2*w3 + w2*w3*exp((w1^2 + w2^2 +
 w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2))]
[ (w3*exp((w1^2 + w2^2 + w3^2)^(1/2)) - w3*exp(-(w1^2 + w2^2 +
 w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)^(1/2)),       (w1*w3*exp(-
(w1^2 + w2^2 + w3^2)^(1/2)) - 2*w1*w3 + w1*w3*exp((w1^2 + w2^2
 + w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)),       (w2*w3*exp(-
(w1^2 + w2^2 + w3^2)^(1/2)) - 2*w2*w3 + w2*w3*exp((w1^2 + w2^2
```

```
+ w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2)), (w3^2*exp(-(w1^2 +
w2^2 + w3^2)^(1/2)) + 2*w1^2 + 2*w2^2 + w3^2*exp((w1^2 + w2^2 +
w3^2)^(1/2)))/(2*(w1^2 + w2^2 + w3^2))]
```

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Question V Overlap Calculations

```
%Overlap of M with Gamma_j is given by c = (1.4) * Tr( M
 (Gamma_J)^(-1) )
G = Gamma_Generator;

result = zeros(1,16);
for i = 1:1:16

    result = result + Gamma_Overlap(conj(G{i}));

end

disp(result)
```

```
  Columns 1 through 13

     1    -1    -1    -1    -1    -1    -1    -1    -1    -1    -1
 1     1

  Columns 14 through 16

     1     1     1
```

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Dependency - Generates Gamma Matrices

```
function result = Gamma_Generator
%Returns a 16 element array of overlap constants with the 16 Gamma
 matrices
%Order is I, gamma_mu, delta_mu_nu, gamma_u*gamma_5, i*gamma_5
%Input is any 4x4 matrix

    %Define commutation
    com = @(A,B) A*B - B*A;

    %Define Pauli matrices
    pauli1 = [0  1 ;
              1  0];
    pauli2 = [0 -i ;
              i  0];
    pauli3 = [1  0 ;
              0 -1];

    %Identity
    I = eye(4,4);

    %Define Gamma matrices
    %Note gamma_0 referenced by 1, gamma_1 by 2 etc
    gamma(3:4,1:2,1) = pauli2;
    gamma(1:2,3:4,1) = pauli2;

    gamma(1:2,1:2,2) = i * pauli3;
    gamma(3:4,3:4,2) = i * pauli3;

    gamma(3:4,1:2,3) =  pauli2;
    gamma(1:2,3:4,3) = -pauli2;

    gamma(1:2,1:2,4) = -i * pauli1;
    gamma(3:4,3:4,4) = -i * pauli1;

    %Delta_mu_nu (mu < nu)
    %Delta_mu_nu = (i/2) * [gamma_mu,gamma_nu]
    for mu = 1:3
        for nu = mu + 1:4
            delta(:,:,mu,nu) = (i/2) * com(gamma(:,:,mu),
 gamma(:,:,nu));
        end
    end

    %The special gamma_5
    gamma_5 = i * gamma(:,:,1)...
                * gamma(:,:,2)...
                * gamma(:,:,3)...
                * gamma(:,:,4);

    %gamm_5 products
```

```
    for x = 1:4
        gamma_5_pro(:,:,x) = gamma(:,:,x) * gamma_5;
    end

    %gamma_5 as a Gamma matrix
    gamma_5_G = i * gamma_5;



    %Define array of Gamma matrices
    Gamma = {

             I,

             gamma(:,:,1),
             gamma(:,:,2),
             gamma(:,:,3),
             gamma(:,:,4),

             delta(:,:,1,2),
             delta(:,:,1,3),
             delta(:,:,1,4),
             delta(:,:,2,3),
             delta(:,:,2,4),
             delta(:,:,3,4),

             gamma_5_pro(:,:,1),
             gamma_5_pro(:,:,2),
             gamma_5_pro(:,:,3),
             gamma_5_pro(:,:,4),

             gamma_5_G

             };

       result = Gamma;
end


ans =

  16×1 cell array

    {4×4 double}
    {4×4 double}
    {4×4 double}
    {4×4 double}
    {4×4 double}
    {4×4 double}
    {4×4 double}
    {4×4 double}
    {4×4 double}
    {4×4 double}
    {4×4 double}
```

```
{4×4 double}
{4×4 double}
{4×4 double}
{4×4 double}
{4×4 double}
```

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Dependency - Calculates Overlaps with Gamma Matrices

```
%Overlap of M with Gamma_j is given by c = (1/4) * Tr( M
 (Gamma_J)^(-1) )
%Returns a 16 element array of overlap constants with the 16 Gamma
 matrices
%Order is I, gamma_mu, delta_mu_nu, gamma_u*gamma_5, i*gamma_5
%Input is any 4x4 matrix

function result = Gamma_Overlap(M)
    %Define commutation
    com = @(A,B) A*B - B*A;

    %Define Pauli matrices
    pauli1 = [0  1 ;
              1  0];
    pauli2 = [0 -i ;
              i  0];
    pauli3 = [1  0 ;
              0 -1];

    %Identity
    I = eye(4,4);

    %Define Gamma matrices
    %Note gamma_0 referenced by 1, gamma_1 by 2 etc
    gamma(3:4,1:2,1) = pauli2;
    gamma(1:2,3:4,1) = pauli2;

    gamma(1:2,1:2,2) = i * pauli3;
    gamma(3:4,3:4,2) = i * pauli3;

    gamma(3:4,1:2,3) =  pauli2;
    gamma(1:2,3:4,3) = -pauli2;

    gamma(1:2,1:2,4) = -i * pauli1;
    gamma(3:4,3:4,4) = -i * pauli1;

    %Delta_mu_nu (mu < nu)
    %Delta_mu_nu = (i/2) * [gamma_mu,gamma_nu]
    for mu = 1:3
        for nu = mu + 1:4
            delta(:,:,mu,nu) = (i/2) * com(gamma(:,:,mu),
 gamma(:,:,nu));
        end
    end

    %The special gamma_5
    gamma_5 = i * gamma(:,:,1)...
                * gamma(:,:,2)...
```

```
                    * gamma(:,:,3)...
                    * gamma(:,:,4);

    %gamm_5 products
    for x = 1:4
        gamma_5_pro(:,:,x) = gamma(:,:,x) * gamma_5;
    end

    %gamma_5 as a Gamma matrix
    gamma_5_G = i * gamma_5;



    %Define array of Gamma matrices
    Gamma = {

             I,

             gamma(:,:,1),
             gamma(:,:,2),
             gamma(:,:,3),
             gamma(:,:,4),

             delta(:,:,1,2),
             delta(:,:,1,3),
             delta(:,:,1,4),
             delta(:,:,2,3),
             delta(:,:,2,4),
             delta(:,:,3,4),

             gamma_5_pro(:,:,1),
             gamma_5_pro(:,:,2),
             gamma_5_pro(:,:,3),
             gamma_5_pro(:,:,4),

             gamma_5_G

             };


     %Compute overlaps
     for x = 1:16
        overlap(x) = (0.25) * trace( M * inv(Gamma{x}) );
     end


    result = overlap;
end
```

Not enough input arguments.

Error in Gamma_Overlap (line 90)
*overlap(x) = (0.25) * trace(M * inv(Gamma{x}));*

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THE UNIVERSITY OF ADELAIDE
DEPARTMENT OF PHYSICS

RELATIVISTIC QUANTUM MECHANICS & PARTICLE PHYSICS

ASSIGNMENT #1:
PROBLEMS TO BE SOLVED INDIVIDUALLY WITHOUT COLLABORATION.

Due: Friday, 8th of May, 2020

1. Show that the Hamiltonian operator

$$H = -i\hbar c\boldsymbol{\alpha}.\boldsymbol{\nabla} + mc^2\beta$$

for the Dirac equation does not commute with orbital angular momentum $\boldsymbol{L}$, and find a spin operator $\boldsymbol{S}$ such that $\boldsymbol{L}+\boldsymbol{S}$ commutes with H. [8 marks]

2. (a) Calculate: $\mathrm{Tr}\{\not{q}\,(\not{p}+m)\,\not{q}\,(\not{p}+m)\}$.
   (b) Write the matrices $\sigma_{\mu\nu}\gamma_5$, $\sigma_{\mu\nu}\sigma^{\mu\nu}$ and $\gamma^\alpha\sigma_{\mu\nu}\gamma_\alpha$ as linear combinations of the sixteen matrices Γ_J discussed in lectures.

[10 marks]

3. Consider two-dimensional space-time, with metric

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (a) Find the most general solution Λ of the Lorentz condition

$$G\Lambda G\Lambda^\top = I\,.$$

   (b) Find all Λ continuously deformable to the identity. Write your answer as exp{real matrix}.
   (c) Construct a two-dimensional representation for gamma matrices.
   (d) Show that the adjoint of a spinor can be constructed as for four-dimensional space time.
   (e) Find a linearly independent set $\{\Gamma_J\}$ which spans spinor space. [14 marks]

4. A body of mass m is at rest. Let

$$\Lambda = \exp -\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}$$

be the Lorentz transformation which boosts the body to four-momentum p^μ without rotating it (in natural units):

$$\Lambda = \frac{1}{m}\begin{pmatrix} p^0 & \vec{p}^\top \\ \vec{p} & mI + (p^0 - m)\hat{p}\hat{p}^\top \end{pmatrix}$$

   (a) Show (e.g. by summing the exponential series) that Λ corresponds to parameters $\omega_{0i} = \eta^i = -\omega_{i0}$ given by

$$\vec{\eta} = \hat{p}\,\ln\left(\frac{p^0 + |\,\vec{p}\,|}{m}\right)$$

   (b) Show that Λ corresponds to a spinor transformation

$$S(\Lambda) = \exp -\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}$$

   which can be written

$$S(\Lambda) = \frac{1}{\sqrt{2m(p^0+m)}}\left(\not{p}\gamma^0 + m\right).$$

   (c) Verify the property

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu_{\ \nu}\gamma^\nu$$

   Hint: Do the cases $\mu = 0$ and $\mu = i$ separately.

[14 marks]

5. Let ψ be a Dirac spinor wave function with charge conjugate ψ_c, and let

$$\{\Gamma_J \,,\, J = 1,\, \dots\,, 16\}$$

be the 16 matrices which span 4×4 spinor space.

Show that $\overline{\psi}_c \Gamma_J \psi_c$ is proportional to $\overline{\psi}\Gamma_J\psi$ for each value of J, and find the constants of proportionality. [6 marks]

6. Consider a Dirac equation of the form

$$\left\{c\boldsymbol{\alpha}.(\boldsymbol{p} - im\omega\boldsymbol{r}\beta) + mc^2\beta\right\}\psi = i\hbar\frac{\partial\psi}{\partial t}$$

where $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla}$ and

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

have their usual meanings.

(a) Show that the Hamiltonian is Hermitean. [2 marks]

(b) In the non-relativistic limit, show that the system reduces to an isotropic harmonic oscillator of angular frequency ω plus a constant $\boldsymbol{L}.\boldsymbol{S}$ coupling. Find the lowest energy eigenvalue. [5 marks]

(c) Solve the relativistic model exactly for all energy eigenvalues in terms of suitable quantum numbers, e.g. the total angular momentum number j. Discuss the degeneracy of the first two energy levels. [5 marks]

Hint: Try the two-spinor decomposition $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, and look up the spherical formulation of the isotropic oscillator in QM books.

Alternatively, a suitable summary of the isotropic harmonic oscillator containing all the pertinent information can be found on the Wolfram website here:

`http://demonstrations.wolfram.com/ThreeDimensionalIsotropicHarmonicOscillator/`