Electrodynamics
Various Problems

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Question I

Part a

The Lienard-Wiechert scalar potential is given by

$$V(r,t) = \frac{kq}{\left(1 - \frac{\vec{v}\cdot\hat{n}}{c}\right) R}$$

where $\vec{v}$ is the particles velocity vector, R is the distance between the particle and our observation position, $\hat{n}$ is the unit displacement vector between the particle and our observation position and k is Coulombs constant. All quantities on the right hand side of the above expression must be evaluated at the retarded time t_r and not the current time t. Begin by finding an expression for the retarded time in terms of the current time and the model variable, b. The standard interpretation of the retarded time gives

$$|\vec{r} - \vec{w}(t)| = c(t - t_r)$$

where $\vec{r}$ is our observation position from the origin and $\vec{w}$ is the particles current position. Evaluating gives

$$\left| \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \sqrt{b^2 - (ct)^2} \\ 0 \\ 0 \end{bmatrix} \right| = c(t - t_r)$$

$$= \pm(x - \sqrt{b^2 - (ct)^2})$$

So

$$t_r = t \pm \frac{1}{c}\left[x - \sqrt{b^2 - (ct)^2}\right]$$

When $t = 0$ we have

$$t_r = \pm\frac{1}{c}(x \pm b)$$

We're given our position is to the right of the charge so, choosing the right direction as the positive $\hat{x}$ direction, we have $x > w(0)$ which implies $x > \pm b$ thus we have

$$(x \pm b) > 0$$

The retarded time must be before the real time to maintain causality. In the case when $t = 0$ this is the same as requiring t_r to be negative so we must only keep the negative sign in the above expression for t_r thus, in full

$$t_r = t - \frac{1}{c}\left[x - \sqrt{b^2 - (ct)^2}\right]$$

The potential at our position, $\vec{r} = x\hat{x}$, depends on the distance between us and the charge position, w, at the retarded time

$$w(t_r) = \sqrt{b^2 - (c(t - \frac{1}{c}(x - \sqrt{b^2 + (ct)^2})))^2}$$

$$= \sqrt{b^2 - (ct - (x - \sqrt{b^2 + (ct)^2}))^2}$$

$$= \sqrt{b^2 - (ct - x + \sqrt{b^2 + (ct)^2})^2}$$

We also require the particle velocity

$$\vec{v(t)} = \frac{d}{dt}w(t)\hat{x}$$

$$= \frac{d}{dt}\sqrt{(b^2 + (ct)^2)}\hat{x}$$

$$= \frac{1}{2}\left(b^2 + (ct)^2\right)^{-\frac{1}{2}}\left(2c^2 t\right)\hat{x}$$

$$= \frac{c^2 t}{w_x(t)}\hat{x}$$

Evaluated at the retarded time

$$v(\vec{t_r}) = \frac{c^2 t_r}{w(t_r)}\hat{x}$$

Where $w(t_r)$ has been derived previously. The Lienard-Wiechert potential is then

$$V(r,t) = \frac{kq}{\left(1 - \frac{\vec{v}\cdot\hat{n}}{c}\right) R}$$

$$V(r,t) = \frac{kq}{\left(1 - \frac{v_x(t_r)}{c}\right)(x - w(t_r))}$$

$$V(r,t) = \frac{kqc}{cx - v(t_r)x - cw(t_r) + v(t_r)w(t_r)}$$

Where $w(t_r)$ and $v_x(t_r) = v(t_r)$ have been previously given. The vector potential is then conveniently given by

$$\vec{A}(r,t) = \frac{\vec{v}(t_r)}{c^2} V(r,t)$$

$$= \frac{t_r}{w(t_r)} V(r,t)$$

Part b

We can find the power radiated by using Lienards generalisation of the Larmor formula

$$P(t) = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left[a(t_r)^2 - |\frac{v(\vec{t_r}) \times a(\vec{t_r})}{c}|^2\right]$$

where $a(t_r)$ is the particles acceleration at the retarded time. As the particles velocity and acceleration are both in the $\hat{x}$ direction the cross product vanishes. The acceleration is given by taking time derivatives of the known velocity

$$\frac{d}{dt}v(t) = \frac{d}{dt}\frac{c^2 t}{w(t)}$$

$$= \frac{c^2}{w(t)} - (w(t))^{-2}c^2 t v(t)$$

$$= \frac{c^2}{w(t)} - (w(t))^{-3}c^4 t^2$$

The power radiated is then

$$P(t) = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left[\frac{c^2}{w(t_r)} - (w(t_r))^{-3}c^4 t^2\right]^2$$

$$= \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left[1 - (w(t_r))^{-2}c^2 t^2\right]^2 \frac{c^4}{w(t_r)^2}$$

We observe the bracketed quantity has a form similar to the γ factor

$$\gamma = \left[1 - \frac{v^2}{c^2}\right]^{-1}$$

$$= \left[1 - \frac{t^2 c^2}{w(t_r)^2}\right]^{-1}$$

Simplifying the power to

$$P = \frac{\mu_0 q^2 \gamma^4 c^3}{6\pi w(t_r)^2}$$

Question II

We see a charge released from rest ($\vec{v}(t=0) = 0$) in the presence of an $\vec{E}$ and $\vec{B}$ field. We wish to find the trajectory of the particle, $\vec{v}(t)$, in a frame where $\vec{E} = 0$. Once found we wish to reverse the transformation to find $v(\vec{t})$ for the initial inertial case. First we must boost from a frame in which $\vec{E} = E_0\hat{z}$ to one in which $\vec{E'} = 0$. We may use the equations for a general Lorentz transformation of fields

$$\vec{E}' = \gamma(\vec{E} + c\vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{E})$$

$$0 = \gamma(E_0\hat{z} + \vec{v} \times B_0\hat{x}) - \frac{\gamma^2}{\gamma+1}\frac{\vec{v}}{c^2}(\vec{v} \cdot E_0\hat{x})$$

So we have

$$\gamma E_0\hat{z} + \gamma B_0\vec{v} \times \hat{x} = \frac{\gamma^2}{\gamma+1}(\vec{v} \cdot \hat{z})E_0\frac{\vec{v}}{c^2}$$

If we set $\vec{v} = v\hat{y}$ we recover

$$\gamma E_0\hat{z} + \gamma B_0 v\hat{y} \times \hat{x} = 0$$

$$\gamma E_0\hat{z} = \gamma B_0\hat{z} = 0$$

So the boost velocity required for the given field conditions is

$$\vec{v} = \frac{E_0}{B_0}\hat{y}$$

This aligns with the provided condition $E_0 < cB_0$ as when substituted into the above velocity expression yields $v < c$.

Given there is now no electric field, the Lorentz force on the particle is

$$\vec{F} = q\vec{v}_{particle} \times \vec{B} = qv_{particle}B\hat{y} \times \hat{z} = qv_{particle}B\hat{x}$$

As the particle begins its motion in the $-\hat{y}$ direction (since this is opposite the boost direction) and the force is orthogonal to the motion, we know the particles speed does not change. It then follows, by symmetry, that the particle is confined to the $x - y$ plane. (Consider a later time, $t_0 + dt$, where the particle has moved to have an infinitesimal velocity in the $\hat{x}$ direction. We can then relabel this time t_0' and rotate axis so the particles velocity is aligned with the $\hat{y}$ axis. This gives the same result as above, thus the particle is confined to the plane.)

The gyradius of a charged particle in the presence of a magnetic field is

$$R = \frac{mv}{qB}$$

And so the particles trajectory is that of a circle in the $x-y$ plane of radius R with its motion beggining in the $-\hat{y}$ direction at a speed equal to the boost velocity.

Returning to the frame containing a non-zero electric field we must reverse the Lorentz boost. As such, the radius of motion will become Lorentz contracted in the $\hat{y}$ direction. Thus the trajectory in the original frame follows an ellipse in the $x-y$ plane with maximum x position of γR and maximum y position of R. The velocity of the particle in the original frame is

$$\begin{bmatrix} v_x' \\ v_y' \\ v_z' \end{bmatrix} = \begin{bmatrix} v_x' \\ v_y \\ 0 \end{bmatrix}$$

where v_y is the particles velocity in the $E=0$ frame and

$$v_x' = \frac{v_x - v_{boost}}{1 - \frac{v_x v_{boost}}{c^2}}$$

where v_x is the velocity in the $E=0$ frame and v_{boost} is the boost velocity.

The resulting motion is such that when v_x is maximised we have $v_x' = 0$ which aligns with our orignal condition that the particle be at rest when located at its starting position.

Question III

Part a

In the frame where the wire is at rest, there are no moving charges and so $\vec{B} = 0$. To find $\vec{E}$, consider Maxwells first equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\lambda}{\epsilon_0}$$

The gradient operator in cylindrical coordinates is given by

$$\vec{\nabla} = \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

By intuition, we know that the $\vec{E}$ field will point only in the $\hat{r}$ direction thus the first Maxwell equation yields

$$\frac{\partial}{\partial r}E = \frac{\lambda}{\epsilon_0}$$

Consider a cylindrical shell of length l enclosing the wire at a distance R. Intuitively the electric field will be parallel to this surface, $\vec{E} \cdot d\vec{a} = Eda$, (excluding the ends of the shell where the dot product vanishes), and will be constant along the surface by symmetry. Thus invoking Gauss' Law gives

$$\int\int\int \vec{\nabla} \cdot \vec{E} dV = \oiint \vec{E} \cdot d\vec{a}$$

$$= E \oiint da$$

$$E(2\pi Rl)$$

We also have

$$\int\int\int \frac{\lambda}{\epsilon_0} dV = \int \frac{q_0}{\epsilon_0} dl = \frac{q_0 l}{\epsilon_0}$$

Thus

$$\frac{q_0 l}{\epsilon_0} = E(2\pi Rl)$$

So

$$\vec{E} = \frac{q_0}{2\pi R\epsilon_0}\hat{r}$$

Choosing R to be an arbitrary distance from the wire provides the general solution

$$\vec{E} = \frac{q_0}{2\pi r\epsilon_0}\hat{r}$$

We can now use the Lorentz transformation equations of the fields to find $\vec{E}$ and $\vec{B}$ in the lab frame.

$$\vec{E}' = \gamma\vec{E} - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{E})$$

$$\vec{B}' = \frac{-\gamma}{c}\vec{\beta} \times \vec{E}$$

Given the wire has velocity $\vec{v} = v\hat{z}$ the magnetic field is then given by

$$\vec{B}' = \frac{-\gamma}{c^2}\frac{vq_0}{2\pi\epsilon_0 r}(\hat{z} \times \hat{r}) = \frac{\gamma v q_0}{2\pi\epsilon_0 r c^2}\hat{\theta}$$

While the electric field becomes

$$\vec{E}' = \frac{\gamma q_0}{2\pi\epsilon_0 r}\hat{r}$$

So we see the primed electric field is equal to the unprimed electric field scaled by a factor of γ.

Part b

In the wires frame of reference the current density is zero while the charge density is q_0. In the lab frame the charge density is length contracted so $q = \gamma q_0$. As the current density is the amount of charge passing a point at a given time, the lab frame sees current density

$$\vec{I} = \vec{v}q = \gamma v q_0 \hat{x}$$

Part c

Using the same method to find the electric field as in the static case, Gauss' law provides

$$\vec{E} = \frac{q}{2\pi\epsilon_0 r}\hat{r} = \frac{\gamma q}{2\pi\epsilon_0 r}\hat{r}$$

which agrees with the result obtained in part(a). To find the magnetic field, consider Stokes theorem

$$\int\int \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \oint \vec{B} \cdot d\vec{l}$$

Consider a circular loop in the plane orthogonal to the wire. By symmetry we know $\vec{B}$ is constant along this loop and also, by intuition (the right hand grip rule), we know $\vec{B} \cdot d\vec{l} = Bdl$ so we have

$$\oint \vec{B} \cdot d\vec{l} = B \oint dl = 2\pi B r$$

So

$$r 2\pi B = \int\int \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \int\int \mu_0 \vec{J} \cdot d\vec{a}$$

$$= \mu_0 |J|$$

$$= \mu_0 \gamma v q_0$$

In summary

$$B = \frac{\gamma v q_0}{2\pi r} \mu_0$$

Using the relation $\sqrt{\frac{\mu_0}{\epsilon_0}} = c$ and the fact the fact the loop considered lies in the $\hat{\theta}$ direction, the magnetic field is

$$\vec{B} = \frac{\gamma v q_0}{2\pi \epsilon_0 c^2} \hat{\theta}$$

Which agrees with part (a).

Question IV

Beginning with the covariant form provided

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

Substituting the values of the field strength tensor explicitly, the covariant expression is then

$$\partial_\mu F^{\mu\nu} = \partial_0 F^{0\nu} + \partial_1 F^{1\nu} + \partial_2 F^{2\nu} + \partial_3 F^{3\nu}$$

$$= \frac{1}{c}\frac{\partial}{\partial t} \begin{bmatrix} 0 \\ \frac{-E_x}{c} \\ \frac{-E_y}{c} \\ \frac{-E_z}{c} \end{bmatrix} - \frac{\partial}{\partial x} \begin{bmatrix} \frac{-E_x}{c} \\ 0 \\ -B_z \\ B_y \end{bmatrix} - \frac{\partial}{\partial y} \begin{bmatrix} \frac{-E_y}{c} \\ B_z \\ 0 \\ -B_x \end{bmatrix} - \frac{\partial}{\partial x} \begin{bmatrix} \frac{-E_z}{c} \\ -B_y \\ B_x \\ 0 \end{bmatrix}$$

$$= \mu_0 \begin{bmatrix} c\rho \\ I_x \\ I_y \\ I_z \end{bmatrix}$$

The $\nu = 0$ case provides the first of Maxwells equations

$$\vec{\nabla} \cdot \vec{E} = \mu_0 c^2 \rho = \frac{\rho}{\epsilon_0}$$

By noting the form of the vector cross product in the entries for ν equal to 1, 2 and 3 we recover

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

which we identify as the fourth of Maxwells equations.

The second covariant expression we are given

$$\partial_\mu G^{\mu\nu} = 0$$

makes use of the dual field strength tensor, G. Again writing the components explicitly we recover a 4-vector relationship

$$\partial_\mu G^{\mu\nu} = \frac{1}{c}\frac{\partial}{\partial t} \begin{bmatrix} 0 \\ -B_x \\ -B_y \\ -B_z \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} -B_x \\ 0 \\ \frac{E_z}{c} \\ \frac{-E_y}{c} \end{bmatrix} + \frac{\partial}{\partial y} \begin{bmatrix} -B_y \\ \frac{-E_z}{c} \\ 0 \\ \frac{E_x}{c} \end{bmatrix} + \frac{\partial}{\partial z} \begin{bmatrix} -B_z \\ \frac{E_y}{c} \\ \frac{E_x}{c} \\ 0 \end{bmatrix}$$

The $\nu = 0$ case recovers the familiar no monopoles law

$$\vec{\nabla} \cdot \vec{B} = 0$$

While identifying the elements of the vector cross product in the ν equals 1,2 and 3 cases recovers Faradays law

$$\frac{-1}{c}\frac{\partial}{\partial t}\vec{B} = \frac{1}{c}\vec{\nabla} \times \vec{E}$$

The second part of question 4 asks us to show the homogeneous Maxwell equations

$$\vec{\nabla} \cdot \vec{B} = 0, \vec{\nabla} \times \vec{E} = 0$$

may be written in the form

$$\partial^\mu F^{\lambda\beta} + \partial^\beta F^{\mu\lambda} + \partial^\lambda F^{\beta\mu} = 0$$

Consider using the identity

$$F^{ab} = \partial^a A^b - \partial^b A^a$$

which gives the expansion

$$\partial^\mu(\partial^\lambda A^\beta - \partial^\beta A^\lambda) + \partial^\beta(\partial^\mu A^\lambda - \partial^\lambda A^\mu) + \partial^\lambda(\partial^\beta A^\mu - \partial^\mu A^\beta)$$

Consider expanding the homogeneous magnetic field Maxwell equation in terms of explicit four potential components

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$$= \partial^1(\partial^2 A^3 - \partial^3 A^2) + \partial^2(\partial^3 A^1 - \partial^1 A^3) + \partial^3(\partial^1 A^2 - \partial^2 A^1)$$

Which we can identify with the expanded three index expression above if we make the index substitution $\mu = 1$,$\lambda = 2$ and $\beta = 3$.

Using the same method as above, consider expanding the remaining homogeneous Maxwell equation in terms of 4-potential entries which yields

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla} V + \frac{\partial}{\partial t}\vec{A})$$

$$= -\vec{\nabla} \times (\vec{\nabla} V) - \vec{\nabla} \times (\frac{\partial}{\partial t}\vec{A})$$

The time derivative term in 4-vector components provides

$$\begin{bmatrix} \partial^2\partial^0 A^3 - \partial^3\partial^0 A^2 \\ \partial^3\partial^0 A^1 - \partial^1\partial^0 A^3 \\ \partial^1\partial^0 A^2 - \partial^2\partial^0 A^1 \end{bmatrix}$$

While the scalar potential term equates to

$$\begin{bmatrix} \partial^2\partial^3 A^0 - \partial^3\partial^2 A^0 \\ \partial^1\partial^3 A^0 - \partial^3\partial^1 A^0 \\ \partial^1\partial^2 A^0 - \partial^2\partial^1 A^0 \end{bmatrix}$$

Finally the term $\frac{-\partial}{\partial t}\vec{B}$ may be written as

$$\begin{bmatrix} \partial^0\partial^2 A^3 - \partial^0\partial^3 A^2 \\ \partial^0\partial^3 A^1 - \partial^0\partial^1 A^3 \\ \partial^0\partial^1 A^2 - \partial^0\partial^2 A^1 \end{bmatrix}$$

Combining all three of the above vectors as dictated by the homogeneous Maxwell equation, and noticing the cyclic nature of the indices, we achieve the final equation

$$\partial^\mu(\partial^\lambda A^\beta - \partial^\beta A^\lambda) + \partial^\beta(\partial^\mu A^\lambda - \partial^\lambda A^\mu) + \partial^\lambda(\partial^\beta A^\mu - \partial^\mu A^\beta)$$

Which matches the original expansion of the 3 index field tensor expression so the second homogeneous Maxwell equation is indeed recovered.

Question V

The equation for a $\vec{B}$ field in a Lorentz transformed frame may be written

$$\vec{B}' = \gamma(\vec{B} - \frac{1}{c}\vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{E})$$

Assume it is possible to have a frame, K, where $\vec{E} = 0$ which when boosted to a frame K' has $\vec{B}' = 0$ then

$$\frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta} \cdot \vec{B}) = \gamma\vec{B}$$

$$(\gamma + 1)\vec{B} = \gamma\vec{\beta}(\vec{\beta} \cdot \vec{B})$$

$$(\gamma + 1)|B|\hat{r_B} = \gamma|\beta|(\vec{\beta} \cdot \vec{B})\hat{r_\beta}$$

This implies that $\hat{r_B} = \hat{r_\beta}$ and so $\vec{\beta} \cdot \vec{B} = |\beta||B|$ thus

$$(\gamma + 1)|B| = \gamma|\beta|^2|B|$$

$$\gamma|B| + |B| = \gamma|B||\beta|^2$$

$$|B| = \gamma(|B||\beta|^2 - |B|)$$

$$(1 - \beta^2)|B| = (\beta^2 - 1)|B|$$

$$(1 - \beta^2)|B| = -(1 - \beta^2)|B|$$

which is only true if either $v = c$ (not possible) or $\vec{B} = 0$ which is a trivial case. Therefore no non-trivial scenarios exist allowing the condition.

An alternate, non-mathematical solution, is as follows. Consider a frame with no E field but a non-zero B field. Assume it is possible to boost to a frame where the current B field vanishes. In doing so, we must increase the magnitude of the E field in order to conserve the observed energy of the fields. This process must also reduce the B field in some direction $\hat{r}$. If this is possible, then it is also possible to continue boosting in this same direction. This would then cause B to become negative and E to further increase. As the energy contained in the fields is irrespective of sign, this would then cause the observed energy in the fields to increase, which violates energy conservation. Thus such a scenario cannot be plausible.

If there is a flaw in the above argument then take the mathematical approach instead.

Question VI

Part a

The Euler-Lagrange equations for the electromagnetic fields are

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial L}{\partial A_\nu} = 0$$

The second quantity is straightforward to calculate

$$\frac{\partial L}{\partial A_\nu} = \frac{\partial}{\partial A_\nu}(-\frac{1}{c}J_\mu A^\mu) = \frac{-1}{c}J_\mu$$

The first quantity gives

$$\partial_a \frac{\partial}{\partial(\partial_a A_b)}\left(\frac{-1}{2}\partial_\mu A_\nu \partial^\mu A^\nu\right)$$

$$= \partial_a \left(\frac{-1}{2}\frac{\partial_\mu A_\nu}{\partial_a A_b}\partial^\mu A^\nu - \frac{1}{2}\partial_\mu A_\nu \frac{\partial^\mu A^\nu}{\partial_a A_b}\right)$$

$$= \partial_a \left(-\frac{1}{2}\partial^a A^b - \frac{1}{2}\partial^a A^b\right)$$

$$= \partial_a \left(-\partial^a A^b\right)$$

Combining the quantities

$$-\partial_a(\partial^a A^b) + \frac{1}{c}J_\mu = 0$$

so

$$\partial_a(\partial^a A^b) = \frac{1}{c}J_\mu$$

Note that

$$A^b = \begin{bmatrix} \frac{V}{c} \\ \vec{A} \end{bmatrix}, J^\mu = \begin{bmatrix} c\rho \\ \vec{I} \end{bmatrix}$$

Element $b = 0$ gives

$$\partial^2 \frac{V}{c} = \frac{1}{c}c\rho$$

$$\frac{1}{c^3}\frac{\partial^2}{\partial t^2}V - \frac{1}{c}\nabla^2 V = \rho$$

While elements $b = 1, 2$ and 3 give

$$\partial^2 \vec{A} = \frac{1}{c}\vec{I}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} = \frac{1}{c}\vec{I}$$

Maxwells equations written in terms of potential quantities give

$$-\nabla^2 - \nabla\frac{\partial}{\partial t}\vec{A} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$$

The third and fourth Maxwell equations combine to give

$$(\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\vec{A}) - \vec{\nabla}(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial}{\partial t}V) = \mu_0\vec{I}$$

We thus see that by choosing to set our equations in the Lorenz gauge, $\vec{\nabla}\cdot\vec{A} = -\mu_0\epsilon_0\frac{\partial}{\partial t}V$, our derived expressions are equivalent to the Maxwell equations.

Part b

We wish to show the difference of the two Lagrangians is equal to $\partial_a\chi^a$, for some four vector χ^a. Note the second term in both Lagrangians, $\frac{-1}{c}J_aA^a$, is equivalent and so cancels in the difference. Thus the problem is reduced to finding

$$\frac{1}{2}\partial_aA_b\partial^aA^b - \frac{1}{4}F_{ab}F^{ab} = \partial_a\chi^a$$

Expanding the field strength tensor in terms of the four vector potential gives

$$\frac{-1}{4}F_{ab}F^{ab} = \frac{-1}{4}(\partial_aA_b - \partial_bA_a)(\partial^aA^b - \partial^bA^a)$$

$$= \frac{-1}{4}(\partial_aA_b\partial^aA^b - \partial_bA_a\partial^aA^b - \partial_bA_a\partial^aA^b + \partial_bA_a\partial^bA^a)$$

$$= \frac{-1}{2}\partial_aA_b\partial^aA^b + \frac{1}{2}\partial_aA_b\partial^bA^a$$

The first term in the above expression acts to cancel the remaining term from the first Lagrangian and so the Lagrangian difference is reduced to

$$= \frac{1}{2}\partial_a A_b \partial^b A^a$$

Which has the form of a four divergence

$$\frac{1}{2}\partial_a (A_b \partial^b A^a)$$

We can now identify the four vector we are looking for as

$$\chi^a = \frac{1}{2} A_b \partial^b A^a$$

In adding a four divergence to a Lagrangian we can find the effect on the action by expanding the definition

$$S = \int_{path} L dt = \iiiint \mathcal{L} d^4 x dt = \int \mathcal{L} d^4 x$$

Adding a four divergence then gives

$$S' = \iiiint \mathcal{L} + \partial_a \chi^a d^4 dt = \int \mathcal{L} d^4 x + \int \partial_a \chi^a d^4 x$$

We may now invoke the 4-D version of Gauss' law to evaluate the difference

$$S' - S = \int \partial_a \chi^a d^4 x$$

$$= \oint_{3-surface} \chi_a d^a x$$

Observe that we require our four vector to be zero infinitely far from any sources and as the integrand bounds are the limits of space, this results in the spatial components of the integrand giving zero. The temporal component, when evaluated at arbitrary beginning and end times, will provide constant terms thus

$$S' - S = constant$$

As the action changes by a constant, we know that the Euler-Lagrange equations - which are derived by considering variations in the action, δS - will be unchanged as $\delta(S' - S) = 0$.

Question VII

We wish to calculate the 4-divergence

$$\partial_\mu \Theta^{\mu\nu} = \partial_\mu g^{\mu\alpha} F_{\alpha\lambda} F^{\lambda\nu} + \frac{1}{4}\partial_\mu g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

First evaluate the quantity $F_{\alpha\beta}F^{\alpha\beta}$ which provides

$$\frac{1}{2}\frac{1}{c^3}\frac{\partial}{\partial t}(E^2 + c^2 B^2)$$

Considering the $\nu = 0$ case, the first term, $\partial_\mu g^{\mu\alpha} F_{\alpha\lambda} F^{\lambda\nu}$, can be expressed as

$$\partial_0 g^{00} F_{0\lambda} F^{\lambda 0} + \partial_1 g^{11} F_{1\lambda} F^{\lambda 0} + \partial_2 g^{22} F_{2\lambda} F^{\lambda 0} + \partial_3 g^{33} F_{3\lambda} F^{\lambda 0}$$

$$= \partial_0 g^{00}(\frac{-1}{c^2}E^2) + \partial_1 g^{11}(\frac{-1}{c}B_z E_y + \frac{1}{c}B_y E_z)$$

$$+\partial_2 g^{22}(\frac{-1}{c}B_x E_z + \frac{1}{c}B_z E_x) + \partial_3 g^{33}(\frac{-1}{c}B_y E_x + \frac{1}{c}B_x E_y)$$

Combining both terms together for the case where $\nu = 0$ provides

$$\frac{1}{c}\partial_t(\frac{-1}{c^2}E^2) - \partial_x(\frac{-1}{c}B_z E_y + \frac{1}{c}B_y E_z) - \partial_y(\frac{-1}{c}B_x E_z + \frac{1}{c}B_z E_x)$$

$$-\partial_z(\frac{-1}{c}B_y E_x + \frac{1}{c}B_x E_y) + \frac{1}{2}\frac{1}{c^3}\frac{\partial}{\partial t}(E^2 + c^2 B^2)$$

Assuming that the 4-divergence is zero, then the above evaluated $\nu = 0$ case recovers

$$\vec{\nabla} \cdot (\vec{B} \times \vec{E}) = \frac{\partial}{\partial t}(\frac{1}{2}\frac{E^2}{c^2} + \frac{1}{2}B^2)$$

Which is the first of the two conservation of Energy momentum equations we require.

The second equation is found by first considering the $\nu = 1$ case of our 4-divergence

$$\partial_0 g^{00} F_{0\lambda} F^{\lambda 1} + \partial_1 g^{11} F_{1\lambda} F^{\lambda 1} + \partial_2 g^{22} F_{2\lambda} F^{\lambda 1} + \partial_3 g^{33} F_{3\lambda} F^{\lambda 1} + \frac{1}{2}\frac{1}{c^3}\frac{\partial}{\partial x}(E^2 + c^2 B^2)$$

Expanding the field strength tensor components gives

$$\frac{1}{c^2}\partial_t(B_y E_z - B_z E_y) - \partial_x \frac{E_x^2}{c^2} - \partial_x(B_z^2 + B_y^2) + \partial_y(B_y B_x - \frac{1}{c^2}E_y E_x)$$

$$+\partial_z(B_z B_x - \frac{1}{c^2}E_z E_x) + \frac{1}{2}\frac{1}{c^3}\partial_x(E^2 + c^2 B^2)$$

Making use of the relationships $c^2 = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$ and $\vec{S} = \vec{E} \times \vec{B}$ this may be simplified to

$$\epsilon_0 \left[(\partial_i E_i)E_x + E_i \partial_i E_x - \frac{1}{2}\partial_x E^2\right] + \frac{1}{\mu_0}\left[(\partial_i B_i)B_x + B_i \partial_i B_x - \frac{1}{2}\partial_x B^2\right] - \frac{1}{c^2}\frac{\partial}{\partial t}S_x$$

We note the first element of the derivative of the Maxwell stress tensor which simplifies the above expression further to

$$(\vec{\nabla} \cdot T)_x - \frac{1}{c^2}\frac{\partial}{\partial t}S_x$$

By noting the cyclic nature of the indices in the original expression of the four divergence it follows similar expressions hold for the y and z components of the above derived x component relationship. Combining all three into a single vector expression provides

$$\vec{\nabla} \cdot T - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{S}$$

Assuming the four divergence is zero then provides the second required Energy momentum conservation equation

$$\vec{\nabla} \cdot T = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{S}$$