Honours General Relativity 2020

Assignment 3

Due date: 13th November 2020 (hand in or email before midnight)

Number of problems: 6 (7 for maths students)

Problem 1

Let S be a generally invariant functional:

$$S = \int d^4x\sqrt{-g}\{\text{scalar}\}$$

of the spacetime metric $g_{\mu\nu}$ and (possibly) matter fields. Consider metric variations $\delta g^{\mu\nu}$ and infinitesimal general coordinate transformations $x \to x' = x + \xi(x)$.

(a) Show that $\delta g^{\mu\nu}$ and ξ^μ are (2,0) and (1,0) tensor fields, respectively.

(b) Show that $(-g)^{-1/2}\delta S/\delta g^{\mu\nu}$ defines a (0,2) tensor.

(c) Consider the special class of metric variations induced by an infinitesimal coordinate transformation. Use the explicit metric connection $\left\{ \begin{array}{c} \alpha \\ \mu\beta \end{array} \right\}$ to deduce the relation:

$$\delta g_{\mu\nu} = -(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) \tag{1}$$

(d) Show that the general invariance of S implies:

$$\nabla_\mu \left((-g)^{-1/2} \frac{\delta S}{\delta g_{\mu\nu}} \right) = 0 \tag{2}$$

Problem 2

On the surface of a two-sphere S^2 of radius a, we have:

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3}$$

Consider the vector $\vec{A}_0 = \vec{e}_\theta$ at $\theta = \theta_0$, $\phi = 0$. The vector is parallel transported all the way around the latitude circle $\theta = \theta_0$ (i.e. over the range $0 \leq \phi \leq 2\pi$ at $\theta = \theta_0$). What is the resulting vector $\vec{A}$? What is its magnitude?

Hint: Derive differential equations for A^θ and A^ϕ as functions of ϕ.

Problem 3

Show that the geodesic equation can be written in the following form:

$$\frac{du_\alpha}{ds} - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} u^\beta u^\gamma = 0 \tag{4}$$

Problem 4

(a) The curved surface of a cylinder can be regarded as a rectangle that has two opposite sides identified with each other, as shown below. The transformation in Figure 1 is a local isometry between the cylinder and the Euclidean plane, and therefore takes geodesics on the cylinder to geodesics on the plane (*i.e.* straight lines).

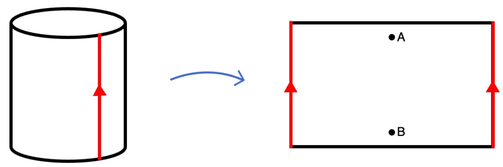

Figure 1: A cylinder cut along its surface and unrolled into a rectangle. The red sides of the rectangle are to be regarded as the same line.

(i) Draw two geodesics connecting points A and B in Figure 1.

(ii) Sketch your geodesics on the normal (*i.e.* "rolled up") cylinder.

(b) Similarly, a local isometry between a cone (not including the vertex) and the plane is shown in Figure 2. This case is of physical relevance: hypothetical objects known as cosmic strings alter the geometry of the cross-sections of space transverse to them into the geometry of a cone, with the string at the vertex. Surprisingly, under general relativity a straight cosmic string would not attract static objects gravitationally (the attraction from its mass is exactly cancelled by the repulsion from its tension), but it would still be detectable from the gravitational lensing around it.

(i) Suppose there were a cosmic string that gives rise to the geometry shown in Figure 2. Indicate on the planar cone in Figure 2 the regions of space where you would see only one image of a distant galaxy in this plane, and where you would see multiple images of the galaxy. Remember light rays follow geodesics. *Hint: without loss of generality, take the distant galaxy to be on the red line.*

(ii) If the geometry were instead as in Figure 3, how many images of the galaxy would you see if you were at point X? *Hint: "glue" multiple copies of the cone together, using the fact that the red lines represent the same points. Straight lines that cross multiple copies of the cone are still geodesics.*

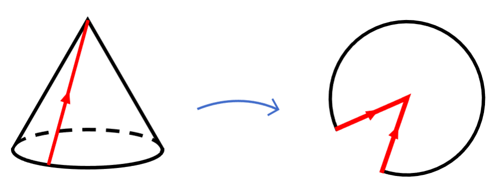

Figure 2: A cone cut along its surface and flattened into a planar shape. The red sides of the shape are to be regarded as the same line.

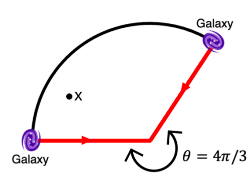

Figure 3: A planar representation of a cone with a deficit angle $\theta = 4\pi/3$, containing a *single* galaxy.

Problem 5

In lectures, we explored the trajectories of massive particles in the Schwarzschild geometry. Now imagine that we have a photon orbiting a spherical massive body where the Schwarzschild solution applies. Assume, for convenience, that the orbit lies in the plane $\theta = \pi/2$.

(a) Starting from the geodesic equations, show that we can write:

$$\bar{E}^2 = \frac{1}{c^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right) \tag{5}$$

where $\bar{E}$ and $\bar{J}$ are constants of the motion and λ is an affine parameter.

(b) Show that there is only one possible circular orbit for a photon, with radius $r = 3GM/c^2$.

(c) Show whether the orbit obtained in part (b) is stable or unstable.

(d) For an observer at a radius of $r = 3GM/c^2$, what is the proper time required for the photon to complete one revolution of the circular orbit?

(e) What is the orbital period measured by a distant observer?

Problem 6

Assume (and don't try to prove) that the invariant interval for every two-dimensional spacetime can be expressed in conformal coordinates:

$$ds^2 = e^{2\phi}(dx^2 - dt^2) \quad \text{where} \quad \phi = \phi(x, t) \tag{6}$$

Calculate the Riemann curvature tensor component R_{txtx}, and write out the 2-dimensional Einstein vacuum equations $R_{ij} = 0$. What is their general solution?

Problem 7

Only do this question if you are a maths student outside of the physics honours stream. You have been warned!

(a) Show that, in two spacetime dimensions ($D = 1+1$), the LHS of the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

vanishes identically. Don't assume Equation (6).

Hint: You will need to look up the 2D form of the Riemann tensor.

(b) Is general relativity possible in a world with $D = 1 + 1$?

Differential Geometry and General Relativity Assignment 3

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Problem I

part a

The equation $(x^\mu)' = (x^\mu) + \epsilon(x^\mu)$ maps from a position within one reference,)x^μ), to a position infinitely close to that position, $(x^\mu)'$. Consider this same mapping, $(x^\mu) \to (x^\mu)'$, from the perspective of an observer in an alternate frame of reference. This observer sees the positions in the original frame Lorenz transformed so $(x^\mu) \to (x^{\mu'})$, $(x^\mu)' \to (x^{\mu'})'$. Note the use of the prime marker to label both frame and coordinate changes, dependent upon its position. The original coordinate deformation is then

$$(x^{\mu'})' = (x^{\mu'}) + \epsilon'(x)$$

where ϵ' is the infinitesimal position update quantity which follows a yet unknown transformation law. Expressing the lorentz property of the new coordinate variables gives

$$(\frac{\partial x^\mu}{\partial x^{\mu'}} x^\mu)' = (\frac{\partial x^\mu}{\partial x^{\mu'}} x^\mu) + \epsilon'(x)$$

Rearranging gives

$$\frac{\partial x^\mu}{\partial x^{\mu'}}((x^\mu)' - (x^\mu)) = \epsilon'(x)$$

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}(\epsilon(x)) = \epsilon'(x)$$

And so $\epsilon(x)$ obeys the transformation law of a rank 1, (1,0) tensor.

Proceed for the metric case in the same manner by considering an infinitesimal shift in an unprimed frame

$$(g^{\mu\nu})' = g^{\mu\nu} + \delta g^{\mu\nu}$$

and a primed frame

$$(g^{\mu'\nu'})' = g^{\mu'\nu'} + \delta g^{\mu'\nu'}$$

Expressing the metric transformation property explicitly gives

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}(g^{\mu\nu})' = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}(g^{\mu\nu}) + \delta g^{\mu'\nu'}$$

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}((g^{\mu\nu})' - g^{\mu\nu}) = \delta g^{\mu'\nu'}$$

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}(\delta g^{\mu\nu}) = \delta g^{\mu'\nu'}$$

So we see $\delta g^{\mu\nu}$ obeys the required law of a rank 2, (2,0) tensor.

part b

In lectures we showed that metric variation of $\sqrt{-g}$ obeys the relation

$$\delta\sqrt{-g} = \frac{-1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$$

Indeed, such an identity is required to preserve the general invariance of S. From this identity it follows that varying the action provided with respect to $g^{\mu\nu}$ has the effect

$$\delta S = \frac{-1}{2}\int d^4x\sqrt{-g}(scalar)(\sqrt{-g}g_{\mu\nu})\delta g^{\mu\nu}$$

Which leads to

$$(\sqrt{-g})^{\frac{-1}{2}}\delta S = \frac{-1}{2}\int d^4x\sqrt{-g}(scalar)(\sqrt{-g}g_{\mu\nu})(\sqrt{-g})^{\frac{-1}{2}}\delta g^{\mu\nu}$$

$$(\sqrt{-g})^{\frac{-1}{2}}\delta S = \frac{-1}{2}\int d^4x\sqrt{-g}(scalar)(g_{\mu\nu})\delta g^{\mu\nu}$$

$$(\sqrt{-g})^{\frac{-1}{2}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{-1}{2}\int d^4x\sqrt{-g}(scalar)(g_{\mu\nu})$$

Thus $(-g)^{-1/2}\delta S\delta g^{\mu\nu}$ obeys the same transformation law as $g_{\mu\nu}$ - that of a rank 2, (0,2) tensor.

part c

Consider expanding the ∇ explicitly in the expression for $\nabla_a\epsilon_b$. This provides

$$\nabla_a\epsilon_b = \partial_a\epsilon_b - \Gamma^c_{ab}\epsilon_c$$

$$= \partial_a\epsilon_b - \frac{1}{2}g^{cd}(\partial_a g_{bd} + \partial_b g_{da} - \partial_d g_{ab})\epsilon_c$$

Taking the sum of $\nabla_a\epsilon_b$ and $\nabla_b\epsilon_a$ then gives

$$\nabla_a\epsilon_b+\nabla_b\epsilon_a = \partial_a\epsilon_b+\partial_b\epsilon_a-\frac{1}{2}g^{cd}(\partial_a g_{bd}+\partial_b g_{da}-\partial_d g_{ab})\epsilon_c-\frac{1}{2}g^{cd}(\partial_b g_{ad}+\partial_a g_{db}-\partial_d g_{ba})\epsilon_c$$

The metric has the property that $g_{ab} = -g_{ba}$. If we make use of this relation we notice all terms provided by the connection are the negatives of one another. Thus, removing these canceling terms provides

$$\nabla_a\epsilon_b + \nabla_b\epsilon_a = \partial_a\epsilon_b + \partial_b\epsilon_a$$

Which, making the identification that the metric provides the variation of coordinates with respect to all other coordinates in the current frame, we may make the identification

$$g_{ab} = -(\partial_a x_b + \partial_b x_a)$$

Considering the infinitesimal case

$$\delta g_{ab} = -(\partial_a \epsilon_b + \partial_b \epsilon_a)$$

And so

$$\delta g_{ab} = -(\nabla_a \epsilon_b + \nabla_b \epsilon_a)$$

As required.

part d

The action, S, is invariant under a change of frame. As shown in part C, the quantity $\delta g_{\mu\nu}$ may be expressed as derivatives of infinitesimal coordinate deformations. As S is invariant under such deformations, the quantity

$$\frac{\delta S}{\delta g_{\mu\nu}}$$

vanishes. Thus we have

$$\nabla_\mu \left((-g)^{-1/2} \frac{\delta S}{\delta g_{\mu\nu}} \right) = \nabla_\mu \left((-g)^{-1/2} 0 \right) = 0$$

As required.

Problem II

The tensor equation for parallel transport is

$$U_a \nabla^a V_b$$

In our two dimensional spacetime this may be written as two equations

$$(U_0 \nabla^0 + U_1 \nabla^1) V_0 = 0$$

$$(U_0 \nabla^0 + U_1 \nabla^1) V_1 = 0$$

The quantity U represents variation of the spacetime coordinates along our path. We are given that θ is held constant, implying the only coordinate change occurs in the variable ϕ. As such it makes sense to parametrise our spacetime path using the variable ϕ. Resulting from this

$$U_0 = \frac{d\theta}{d\phi} = 0$$

$$U_1 = \frac{d\phi}{d\phi} = 1$$

Parallel transport is therefore specified by the conditions

$$\nabla^1 V_0 = 0$$
$$\nabla^1 V_1 = 0$$

Expanding the covariant derivative

$$\frac{d}{d\phi} V_1 = \Gamma^1_{21} V_1 + \Gamma^2_{21} V_2$$

$$\frac{d}{d\phi} V_2 = \Gamma^1_{22} V_1 + \Gamma^2_{22} V_2$$

Using the provided line element and the unitary condition of the multiplication of two metrics we deduce the tensor elements

$$g_{\mu\nu} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{bmatrix}$$

$$g^{\mu\nu} = \begin{bmatrix} a^{-2} & 0 \\ 0 & a^{-2}(\sin\theta)^{-1} \end{bmatrix}$$

We may use the explicit relation for the connection coefficients in unison with the metric identities to write

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})$$

Making use of the fact all off diagonal elements of the metric vanish to simplify our algebra, we have

$$\Gamma^1_{21} = \frac{1}{2} a^{-2}(\partial_\phi a^2)$$

$$\Gamma^2_{21} = \frac{1}{2} a^{-2}(\sin\theta)^{-1}(\partial_\theta a^2 \sin^2\theta)$$
$$= \frac{1}{2}(\sin\theta)^{-1}(2\sin\theta\cos\theta)$$

$$= \frac{\cos\theta}{\sin\theta}$$
$$= (\tan\theta)^{-1}$$

$$\Gamma^1_{22} = \frac{1}{2}a^{-2}(-\partial_\theta a^2 \sin^2\theta)$$
$$= -\frac{1}{2}(a^2 2\sin\theta\cos\theta)$$
$$= -(\sin\theta\cos\theta)$$

$$\Gamma^2_{22} = \frac{1}{2}a^{-2}(\sin\theta)^{-1}(\partial_\phi g_{22} + \partial_\phi g_{22} - \partial_\phi g_{22})$$
$$= 0$$

We have now arrived at two differential equations describing how a tensor V changes during the process of parallel transport. In full, these equations are

$$\partial_\phi V_1 = (\tan\theta)^{-1} V_2$$

$$\partial_\phi V_2 = -(\sin\theta\cos\theta) V_1$$

To solve for V, first take second derivatives

$$\frac{\partial^2}{\partial\phi^2} V_1 = (\tan\theta)^{-1}\partial_\phi V_2$$

$$= -\frac{\cos\theta}{\sin\theta}\sin\theta\cos\theta V_2$$

$$= (\cos\theta)^2 V_2$$

By inspection we see a solution to this equation is

$$V_1 = A\exp^{i(\cos\theta)\phi} + B\exp^{-i(\cos\theta)\phi}$$

Repeating the derivation for the second parallel transport equation recovers

$$V_2 = C\exp^{i(\cos\theta)\phi} + D\exp^{-i(\cos\theta)\phi}$$

Due to the coupling of our equations, the model parameters are reduced by 2 as

$$\partial_\phi V_1 = -(\tan\theta)^{-1} V_2$$

$$\cos\theta(A\exp^{i(\cos\theta)\phi} - B\exp^{-i(\cos\theta)\phi}) = \frac{\cos\theta}{\sin\theta}(C\exp^{i(\cos\theta)\phi} + D\exp^{-i(\cos\theta)\phi})$$

$$(\sin\theta A\exp^{i(\cos\theta)\phi} - \sin\theta B\exp^{-i(\cos\theta)\phi}) = (C\exp^{i(\cos\theta)\phi} + D\exp^{-i(\cos\theta)\phi})$$

Equating coefficients reveals

$$C = (\sin\theta)A$$
$$D = -(\sin\theta)B$$

Which allows us to write

$$V_2 = (\sin\theta)A\exp^{i(\cos\theta)\phi} - (\sin\theta)B\exp^{-i(\cos\theta)\phi}$$

As a check for consistency observe our original differential equations coupling the elements of V are obeyed by the above identities. The initial conditions we are provided are $\phi_i = 0$ and $\theta_i = \theta_0$. Substitution into our vector identities leads to

$$V_1^i = \theta_0 = A + B$$

$$V_2^i = 0 = A\sin\theta_0 - B\cos\theta_0$$

Simultaneous solutions of the above provide

$$A = B = \frac{\theta_0}{2}$$

The imposed constraints lead our particular solutions to be

$$V_1 = \frac{\theta_0}{2}\exp^{i(\cos\theta)\phi} + \frac{\theta_0}{2}\exp^{-i(\cos\theta)\phi}$$

$$V_2 = (\sin\theta)\frac{\theta_0}{2}\exp^{i(\cos\theta)\phi} - (\sin\theta)\frac{\theta_0}{2}\exp^{-i(\cos\theta)\phi}$$

Equivalently using trigonometric identities

$$V_1 = \frac{\theta_0}{2}\sin((\cos\theta)\phi) + \frac{\theta_0}{2}\cos((\cos\theta)\phi)$$

$$V_2 = (\sin\theta)\frac{\theta_0}{2}\sin((\cos\theta)\phi) - (\sin\theta)\frac{\theta_0}{2}\cos((\cos\theta)\phi)$$

After transporting the vector from $\phi = 0$ to $\phi = 2\pi$ the value of V is given by the above equations with $\theta = \theta_0$ and $\phi = 2\pi$. The length is given by

$$V_a V^a = V^a g_{ab} V^b$$

Which, in matrix notation, is equivalent to

$$V^T g V = a^2 V_1^2 + a^2 \sin^2\theta V_2^2$$

Substituting our identities for V_1 and V_2 provides

$$\frac{\theta_0^2}{4}a^2(s^2 + c^2 + 2sc) + \frac{\theta_0^2}{4}a^2\sin^4\theta_0(s^2 + c^2 - 2sc)$$

Where we have made the helpful substitution

$$s = \sin(\cos(\theta_0)\phi)$$

$$c = \cos(\cos(\theta_0)\phi)$$

Continuing the expansion provides

$$\frac{\theta_0^2}{4}a^2(1 + 2sc) + \frac{\theta_0^2}{4}a^2\sin^4\theta_0(1 - 2sc)$$

$$= \frac{\theta_0^2}{4}a^2(1 + 2sc + \sin^4\theta_0 - 2sc\sin^4\theta_0)$$

Replacing our adopted shorthand with their original forms and specifying $\phi = 2\pi$ gives the final result

$$= \frac{\theta_0^2}{4}a^2(1 + 2\sin(\cos(\theta_0)2\pi)\cos(\cos(\theta_0)2\pi) + \sin^4\theta_0$$
$$-2\sin(\cos(\theta_0)2\pi)\cos(\cos(\theta_0)2\pi)\sin^4\theta_0)$$

The process of parallel transport acts to move a tensor along a space-time manifold whilst keeping its original length intact. This is achieved by compensating for the coordinate deformation with the metric connection. By this logic, the length derived above should be equal to the original length. As this is not so, there must be an error somewhere in the derivation.

Problem III

The geodesic equation is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

Expressing the Christoffel symbol explicitly in the Geodesic equation, assuming an affine parametrisation of space-time curves, gives

$$\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}g^{\mu a}(\partial_\alpha g_{\beta a} + \partial_\beta g_{a\alpha} - \partial_a g_{\alpha\beta})\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

Continuing the expansion and making use of the relation $\frac{\partial}{\partial x^\epsilon}\frac{dx^\epsilon}{d\tau} = \frac{\partial}{\partial\tau}$ gives

$$\frac{d}{d\tau}\frac{dx^\mu}{d\tau} + \frac{1}{2}g^{\mu a}\frac{\partial}{\partial\tau}g_{\beta a}\frac{dx^\beta}{d\tau} + \frac{1}{2}g^{\mu a}\frac{\partial}{\partial\tau}g_{a\alpha}\frac{dx^\alpha}{d\tau} - \frac{1}{2}g^{\mu a}\frac{\partial}{\partial x^a}g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

Premultiplying by $g_{\delta\mu}$ provides

$$\frac{d}{d\tau}\frac{dx^\mu}{d\tau} + \frac{1}{2}\delta^a_\delta\frac{\partial}{\partial\tau}g_{\beta a}\frac{dx^\beta}{d\tau} + \frac{1}{2}\delta^a_\delta\frac{\partial}{\partial\tau}g_{a\alpha}\frac{dx^\alpha}{d\tau} - \frac{1}{2}\delta^a_\delta\frac{\partial}{\partial x^a}g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

$$= \frac{d}{d\tau}\frac{dx^\mu}{d\tau} + \frac{1}{2}\frac{\partial}{\partial\tau}g_{\beta\delta}\frac{dx^\beta}{d\tau} + \frac{1}{2}\frac{\partial}{\partial\tau}g_{\delta\alpha}\frac{dx^\alpha}{d\tau} - \frac{1}{2}\frac{\partial}{\partial x^\delta}g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

Identifying the four velocity as $\frac{d}{d\tau}\frac{dx^\mu}{d\tau} = \frac{d}{d\tau}u_\alpha$ gives

$$\frac{d}{d\tau}u_\alpha - \frac{1}{2}\frac{\partial}{\partial x^\delta}g_{\alpha\beta}u^\beta u^\alpha + \frac{1}{2}(\frac{\partial}{\partial\tau}g_{\beta\delta}\frac{dx^\beta}{d\tau} + \frac{\partial}{\partial\tau}g_{\delta\alpha}\frac{dx^\alpha}{d\tau})$$

The symmetric property of the metric tensor provides $g_{\alpha\beta} = -g_{\beta\alpha}$ so we have

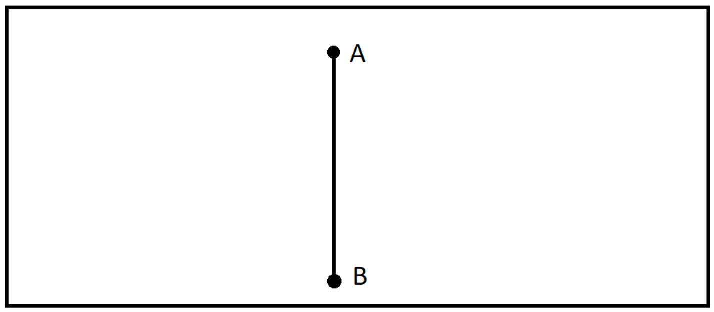
$$\frac{du_\alpha}{d\tau} - \frac{1}{2}\frac{\partial}{\partial x^\delta}g_{\alpha\beta}u^\beta u^\alpha = 0$$

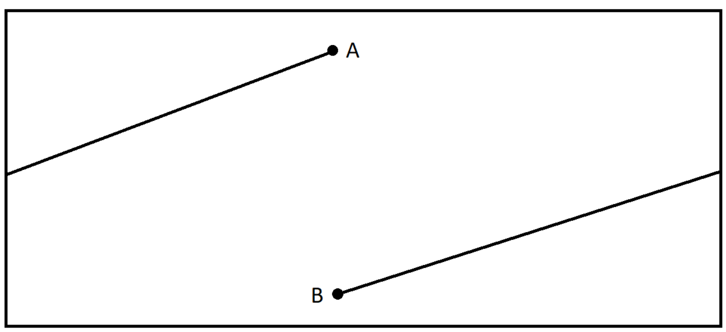
Which is the result we desire.

Problem IV

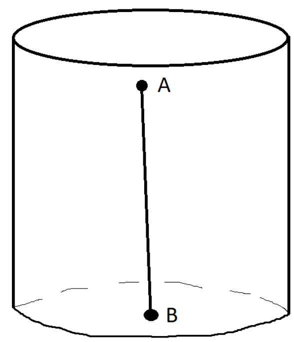
Part (a)

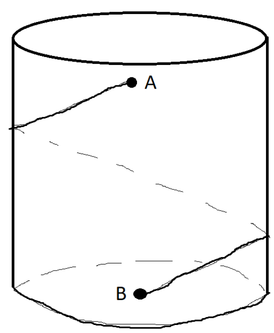
i





ii



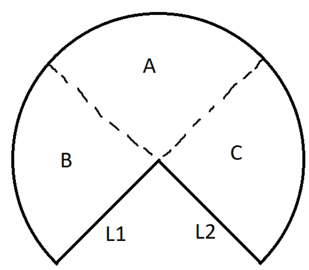


Part (b)

i

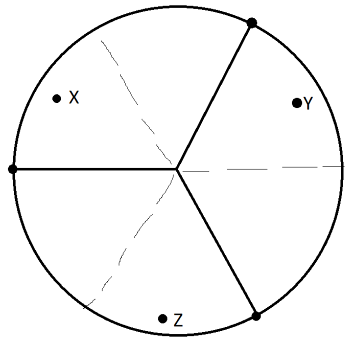
A galaxy laying along the lines marking the incision of the circle (lines L1 and L2 in the provided schematic) will emit light rays in all directions. For the galaxy on line L1, this image will be visible in the region $I_1 = A \cup B$. Similarly the same effect occurs for the galaxy when taken to be on line L2. In this case the image is visible in section $I_2 = A \cup C$. The region providing multiple images is then the intersection of regions I_1 and I_2

$$I_1 \cap I_2 = A$$



ii

In this geometry, the position of the galaxies means that geodesics drawn from them will reach every point on the planar surface. As a result, there are 3 images of the galaxy visible at point X, as labeled on the provided diagram. Images incident to points Y and Z arrive at equivalent angles to those at position X and as such are taken to be the same image and not additional images.



Problem V

The first integral of the Geodesic equation is

$$g_{\mu\nu} d\dot{x}^{\mu} d\dot{x}^{\nu} = L$$

For a photon, this must correspond to a light-like world line, thus

$$L = 0$$

Identifying L as the lagrangin and writing explicitly for a schwarzchild geometry gives

$$L = c^2 \left(1 - \frac{2\mu}{r}\right) \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2 \left(\dot{\theta}^2 + sin^2\theta\dot{\phi}^2\right) = 0$$

Assuming the particle to be confined to the $\theta = \frac{\pi}{2}$ plane

$$L = c^2 \left(1 - \frac{2\mu}{r}\right) \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\phi}^2 = 0$$

We know (from lectures) that two solutions of the geodesic equation for a general worldline in a schwarzchild geometry are

$$\left(1 - \frac{2\mu}{r}\right) \dot{t} = k$$

$$r^2\dot{\phi} = h$$

Simplifying our lagrangian to

$$L = c^2\left(1-\frac{2\mu}{r}\right)^{-1}k^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - \frac{h^2}{r^2} = 0$$

Rearranging for k^2 gives

$$c^2k^2 = \dot{r}^2 + \frac{h^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)$$

Noting the use of the relation $\mu = \frac{GM}{c^2}$. Further, we may write

$$k^2 = \frac{1}{c^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{h^2}{c^2r^2}\left(1-\frac{2GM}{c^2r}\right)$$

Using the identity $\left(\frac{dr}{d\lambda}\right) = \dot{r}$ for a suitable affine parameter λ. We then identify $k^2 = \bar{E}^2$ and $h^2 = \bar{J}^2$ as the constants of motion referenced in the assignment sheet.

b

Begin with $h = r^2\dot{\phi}$ to write

$$\frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} = \frac{h}{r^2}\frac{dr}{d\phi}$$

Now substitute into our energy equation found in (a) to achieve

$$c^2k^2 = \left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} - \frac{2\mu h^2}{r^3}$$

Now, allow the mapping $\frac{1}{r} \to u$ which provides

$$c^2k^2 = \left(hu^2\frac{dr}{d\phi}\right)^2 + h^2u^2 - 2\mu h^2u^3$$

Making use of

$$\frac{dr}{d\phi} = \frac{d}{d\phi}\left(\frac{1}{u}\right) = -u^{-2}\frac{du}{d\phi}$$

we may write

$$c^2k^2 = h^2\left(\frac{du}{d\phi}\right)^2 + h^2u^2 - 2\mu h^2u^3$$

Differentiating with respect to ϕ and dividing by h^2 gives

$$0 = \frac{d^2u}{d\phi^2} + u - 3\mu u^2$$

Considering circular orbits, we know that $u = \frac{1}{r}$ does not change thus all derivatives of u vanish which leads us to

$$u = 3\mu u^2$$

Which has one non-trivial solution

$$\frac{1}{u} = r = 3\mu$$

Identifying $\mu = \frac{GM}{c^2}$ gives our desired result

$$r = \frac{3GM}{c^2}$$

part c

The Newtonian energy equation for a photon in an orbit is

$$E = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V_{eff}(r)$$

Comparing this with our energy equation derived in (a) we can identify $\frac{1}{2}k^2c^2$ as the energy and the effective potential as being

$$v_{eff}(r) = \frac{1}{2}\left(\frac{h}{r}\right)^2 - \left(\frac{h}{r}\right)^2 \frac{\mu}{r}$$

Stable and unstable orbits correspond to extrema of the effective potential. We may find such points by considering derivatives with respect to the co-ordinate r

$$\frac{dv_{eff}(r)}{dr} = 3h^2\mu r^{-4} - h^2 r^{-3}$$

Setting this to zero for the case of extrema and solving provides

$$3\mu r^{-4} - r^{-3} = 0$$

$$3\mu - r = 0$$

$$r = \frac{3GM}{c^2}$$

So we see there is only one extremum. Note that this matches the result found in part (a) which provides confidence we are on the right path.

Taking a second derivative to assess stability gives

$$\frac{d}{dr}\left(\mu r^{-4} - r^{-3}\right) = 0$$

$$-12\mu r^{-5} + 3r^{-4} = 0$$

Multiplying through by r and solving gives

$$r = 4\mu$$

This is the turning point of the second derivative. If r is greater than this value then the second derivative will be positive, corresponding to a stable orbit. In our case we have

$$r_{circle} = 2\mu$$

$$r_{turn} = 4\mu$$

Thus

$$r_{circle} < r_{turn}$$

And the orbit is unstable.

part d

Consider using the product rule

$$\frac{d\phi}{dt} = \left(\frac{d\lambda}{dt}\frac{d\phi}{d\lambda}\right)$$

Where λ is the affine parameter used thus far. Using the general solutions of the Schwarzchild Geodesic equations provided previously, we then have

$$\frac{d\phi}{dt} = \left(1 - \frac{2\mu}{r}\right) k^{-1} \frac{h}{r^2}$$

Now consider another previously derived result

$$\frac{c^2k^2}{h^2} = \left(\frac{du}{d\phi}\right)^2 + u^2 - 3\mu u^3$$

Fixing u to be constant gives

$$\frac{c^2k^2}{h^2} = u^2\left(1 - 3\mu u\right)$$

In terms of r

$$\frac{c^2k^2}{h^2} = \frac{1}{r^2}\left(1 - \frac{3\mu}{r}\right)$$

And so the ration $\frac{k}{h}$ is given by

$$\frac{k}{h} = \frac{1}{cr}\left(1 - \frac{3\mu}{r}\right)^{\frac{1}{2}}$$

Combining this with our orbital period equation produces the relation

$$\frac{d\phi}{dt} = \left(1 - \frac{2\mu}{r}\right) k^{-1} \frac{h}{r^2}$$

$$= \left(1 - \frac{2\mu}{r}\right) \frac{1}{r^2} cr \left(1 - \frac{3\mu}{r}\right)^{\frac{-1}{2}}$$

$$= \frac{c}{r} \frac{\left(1 - \frac{2\mu}{r}\right)}{\left(1 - \frac{3\mu}{r}\right)^{\frac{1}{2}}}$$

For an observer at a position of $r_0 = 3\mu$ we may substitute r_0 into the above expression to find the observed orbital period. In doing so we have the result

$$\frac{d\phi}{dt} \to \infty$$

Which implies one orbit is completed instantaneously. This seems problematic yet such results are often unintuitive in GR.

part e

Continuing our working in part d, we may find the orbital velocity of an observer positioned infinitely far from the mass by considering our equation in the regime $r \to \infty$. In doing so we have the result

$$\frac{d\phi}{dt} \to 0$$

Which corresponds to an infinitely long orbital period. This also appears to imply the photon is observed to have an angular velocity of 0 which again seems problematic, but by the above reasoning, may be a counter intuitive occurrence caused by GR.

Problem VI

The definition of the Riemann tensor provides

$$R_{txtx} = \frac{1}{2}(\partial_x\partial_t g_{xt} - \partial_x\partial_x g_{tt} + \partial_t\partial_x g_{tx} - \partial_t\partial_t g_{xx}) - g^{ef}(\Gamma_{ett}\Gamma_{fxx} - \Gamma_{etx}\Gamma_{fxt})$$

To find the value of the metric tensor recall the relation

$$ds^2 = dx^\mu dx^\nu g_{\mu\nu}$$

Which, in our case, is given by

$$e^{2\phi}dx^2 - e^{2\phi}dt^2$$

Providing a metric

$$g_{\mu\nu} = \begin{bmatrix} e^{2\phi} & 0 \\ 0 & -e^{2\phi} \end{bmatrix}$$

The partial derivative term of the Riemann curvature tensor is then

$$\frac{1}{2}(-\partial_x\partial_x(-e^{2\phi}) - \partial_t\partial_t(e^{2\phi}))$$

$$= e^{2\phi}(\partial_x^2\phi + 2(\partial_x\phi)^2 - 2(\partial_t\phi)^2 - \partial_t^2\phi$$

The connection coefficients we require are

$$\Gamma_{abc} = \frac{1}{2}(\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc})$$

Which in our two dimensional spacetime are

$$\Gamma_{xxx} = \frac{1}{2}(\partial_x g_{xx}) = \phi_x e^{2\phi}$$

$$\Gamma_{xxt} = \frac{1}{2}(\partial_t g_{xx}) = \phi_t e^{2\phi}$$

$$\Gamma_{xtx} = \frac{1}{2}(\partial_t g_{xx}) = \phi_t e^{2\phi}$$

$$\Gamma_{txx} = \frac{1}{2}(-\partial_t g_{xx}) = -\phi_x e^{2\phi}$$

Noticing the metric tensor property $g_x x = -g_t t$ allows us to multiply the above expressions by -1, while interchanging x and t, to obtain the alternate coordinate connection coefficients

$$\Gamma_{ttt} = -\phi_t e^{2\phi}$$

$$\Gamma_{ttx} = -\phi_x e^{2\phi}$$

$$\Gamma_{txt} = -\phi_x e^{2\phi}$$

$$\Gamma_{xtt} = \phi_x e^{2\phi}$$

Recalling the product of two metrics must recover the identity matrix, we have the identity for the contravariant metric

$$g^{ef} = \begin{bmatrix} e^{-2\phi} & 0 \\ 0 & -e^{-2\phi} \end{bmatrix}$$

So the connection term of the curvature tensor is then

$$g^{xx}(\Gamma_{xtt}\Gamma_{xxx} - \Gamma_{xtx}\Gamma_{xxt}) + g^{xt}(\Gamma_{xtt}\Gamma_{txx} - \Gamma_{xtx}\Gamma_{xt}) + g^{tx}(\Gamma_{ttt}\Gamma_{xxx} - \Gamma_{ttx}\Gamma_{xxt}) + g^{tt}(\Gamma_{ttt}\Gamma_{txx} - \Gamma_{ttx}\Gamma_{txt})$$

$$= g^{xx}(\Gamma_{xtt}\Gamma_{xxx} - \Gamma_{xtx}\Gamma_{xxt}) + g^{tt}(\Gamma_{ttt}\Gamma_{txx} - \Gamma_{ttx}\Gamma_{txt})$$

$$= e^{-2\phi}(\phi_x e^{2\phi}\phi_x e^{2\phi} - \phi_t e^{2\phi}\phi_t e^{2\phi}) - e^{-2\phi}(\phi_t e^{2\phi}\phi_t e^{2\phi} - \phi_x e^{2\phi}\phi_x e^{2\phi})$$

$$2e^{2\phi}(\phi_x^2 - \phi_t^2)$$

We can now combine all our working to arrive at a single expression for the Riemann curvature tensor in our two dimensional spacetime.

$$R_{txtx} = e^{2\phi}\left[\partial_x^2\phi - \partial_t^2\phi + 2(\partial_x\phi)^2 - 2(\partial_t\phi)^2 + 2(\partial_t\phi)^2 - 2(\partial_x\phi)^2\right]$$

$$= e^{2\phi}\left[\partial_x^2\phi - \partial_t^2\phi\right]$$

The Einstein field equations in a vacuum are given by the condition $R_{ab} = 0$. This tensor may alternatively be written as

$$R_{ab} = g^{ce}R_{eabc} = -g^{ce}R_{aebc}$$

This gives us 4 possible elements. The first four quantities we will need to evaluate these elements are

$$R_{xtxt}, R_{ttxt}, R_{xttt}, R_{tttt},$$

The final 3 quantities are zero by the symmetry properties of the Riemann tensor as

$$R_{tttt} = -R_{tttt}$$

$$R_{xttt} = -R_{xttt}$$

$$R_{ttxt} = -R_{ttxt}$$

We also need expressions for

$$R_{txtx}, R_{xxtx}, R_{txxx}, R_{xxxx},$$

Again, the final three of which are zero by symmetry. Consider using the curvature tensor property

$$R_{abcd} = -R_{abdc}, R_{abcd} = -R_{bacd}$$

twice in succession which gives

$$R_{abcd} = R_{badc}$$

and thus

$$R_{txtx} = R_{xtxt}$$

These results show the only non-trivial equation we must consider involves our previously calculated Riemann tensor element. This equation, in a vacuum, is

$$R_{xx} = g^{tt} R_{xtxt} = -e^{2\phi} e^{2\phi} \left[\partial_x^2 \phi - \partial_t^2 \phi \right] = 0$$

$$\partial_x^2 \phi = \partial_t^2 \phi$$

By inspection, the general solution to this equation is

$$\phi = A e^{k(x+t)} + B e^{-k(x+t)}$$

where the constant k may very well be complex. Indeed, if it is, the result is far more well behaved.