Differential Geometry and General Relativity
Assignment 2

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Question 1

Part a

The directional derivative gives the rate of change of a function f along a curve C_i. In three dimensional cartesian coordinates this is given by

$$\frac{df}{dt} = \frac{dx}{dt}\frac{df}{dx} + \frac{dy}{dt}\frac{df}{dy} + \frac{dz}{dt}\frac{df}{dz}$$

For curve C_1

$$\frac{dx}{dt} = 0, \frac{dy}{dt} = 1, \frac{dz}{dt} = 1$$

$$\frac{df}{dx} = 2xz, \frac{df}{dy} = 2yz, \frac{df}{dz} = x^2 + y^2$$

So the directional derivative is

$$v_1 = 2yz + x^2 + y^2$$

Evaluated at the point p

$$v_1(p) = 1$$

For curve C_2

$$\frac{dx}{dt} = -sinh(t), \frac{dy}{dt} = cosh(t), \frac{dz}{dt} = 1 + 2t$$
$$\frac{df}{dx} = 2xz, \frac{df}{dy} = 2yz, \frac{df}{dz} = x^2 + y^2$$

So the directional derivative is

$$v_1 = -2xzsinh(t) + 2yzcosh(t) + x^2 + 2tx^2 + 2ty^2 + y^2$$

Using the definition of the first curve

$$C_1(t) = (x, y, z) = (1, t, t)$$

We see the point $p = (1, 0, 0)$ corresponds to a t value of 0. The directional derivative of C_2 evaluated at p is then

$$v_1(p) = 1$$

Part b

We wish to find the tangent vectors in the coordinate basis

$$(\partial_x, \partial_y, \partial_z)$$

This is the rate of change of x, y and z along each point in the curve normalized to unit length. This may be written

$$(\frac{\partial x_c}{\partial t}, \frac{\partial y_c}{\partial t}, \frac{\partial z_c}{\partial t})$$

For curve coordinates (x_c, y_c, z_c). Using the results from part a for C_1 this gives

$$(\frac{\partial x_c}{\partial t}, \frac{\partial y_c}{\partial t}, \frac{\partial z_c}{\partial t}) = (0, 1, 1)$$

Normalizing to unit length we have

$$v = \frac{1}{\sqrt{2}}(0, 1, 1)$$

To recover the original length of the vector find the length in cartesians

$$\sqrt{\left(\frac{\partial x_c}{\partial t}\right)^2 + \left(\frac{\partial y_c}{\partial t}\right)^2 + \left(\frac{\partial z_c}{\partial t}\right)^2} = \sqrt{1+1} = \sqrt{2}$$

And so the equivalent length vector in the derivative basis is

$$D = (\sqrt{2}, \sqrt{2}, \sqrt{2})$$

Interestingly, though the first element of this vector is $\sqrt{2}$, because the derivative in this direction is zero, the total length is preserved

$$\frac{1}{\sqrt{2}} = \sqrt{(D_1\partial_x)^2 + (D_1\partial_y)^2 + (D_1\partial_z)^2} = \sqrt{\left(\sqrt{2}\frac{1}{\sqrt{2}}\right)^2 + \left(\sqrt{2}\frac{1}{\sqrt{2}}\right)^2} = \sqrt{2}$$

As the tangent vector to the curve C_2 at the point p is the same as C_1 and the curve derivatives $(\partial_x, \partial_y, \partial_z)$ are also equal, the resulting vector in the derivative basis is equivalent to the C_1 case.

Question 2

Part a

Begin by writing our equation in matrix notation

$$f = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A rotation results from

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Replacing x and y in our original equation by the rotated equivalent and simplifying via MATLAB we achieve

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 2 - sin(2T) & cos(2T) \\ cos(2T) & sin(2T) + 2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$2x'^2 - sin(2T)x'^2 + cos(2T)y'x' + cos(2T)y'x' + y'^2 sin(2T) + 2y'^2$$

Choosing $T = \frac{\pi}{4}$

$$2x'^2 - x'^2 + y'^2 + 2y'^2$$

$$= x'^2 + 3y'^2 = 1$$

As required

Part b

The components of the tangent vector (∂_x, ∂_y) are

$$\frac{dx}{d\psi} = \frac{1}{\sqrt{2}}\left(-sin\psi + \frac{1}{\sqrt{3}}cos\psi\right)$$
$$\frac{dy}{d\psi} = \frac{1}{\sqrt{2}}\left(sin\psi + \frac{1}{\sqrt{3}}cos\psi\right)$$

Now for $v(f) = v(xy)$

$$\frac{dx}{d\psi}\frac{d}{dx}(xy) + \frac{dy}{d\psi}\frac{d}{dy}(xy)$$

$$= \frac{1}{\sqrt{2}}\left(-sin\psi + \frac{1}{\sqrt{3}}cos\psi\right)\frac{1}{\sqrt{2}}\left(-cos\psi + \frac{1}{\sqrt{3}}sin\psi\right)$$

$$+\frac{1}{\sqrt{2}}\left(sin\psi + \frac{1}{\sqrt{3}}cos\psi\right)\frac{1}{\sqrt{2}}\left(cos\psi + \frac{1}{\sqrt{3}}sin\psi\right)$$

$$= \frac{1}{2}\left(sin\psi cos\psi + \frac{1}{3}cos\psi sin\psi - \frac{1}{\sqrt{3}}sin^2\psi - \frac{1}{\sqrt{3}cos^2\psi}\right)$$

$$+\frac{1}{2}\left(sin\psi cos\psi + \frac{1}{3}cos\psi sin\psi + \frac{1}{\sqrt{3}}sin^2\psi + \frac{1}{\sqrt{3}cos^2\psi}\right)$$

$$= \frac{4}{3}sin\psi cos\psi$$

Now consider expanding

$$\frac{2}{\sqrt{3}}(x^2 - y^2)$$

To get

$$\frac{2}{\sqrt{3}}\left(\left(\frac{1}{\sqrt{2}}cos\psi+\frac{1}{\sqrt{6}}sin\psi\right)^2-\left(-\frac{1}{\sqrt{2}}cos\psi+\frac{1}{\sqrt{6}}sin\psi\right)^2\right)$$

$$=\frac{2}{\sqrt{3}}(\frac{1}{2}cos^2\psi+\frac{1}{6}sin^2\psi+\frac{2}{\sqrt{12}}sin\psi cos\psi$$

$$-\frac{1}{2}cos^2\psi-\frac{1}{6}sin^2\psi+\frac{2}{\sqrt{12}}sin\psi cos\psi)$$

$$=\frac{2}{\sqrt{3}}\left(\frac{4}{\sqrt{12}}sin\psi cos\psi\right)$$

$$=\frac{4}{3}cos\psi sin\psi$$

We see the result is equivalent to that derived above and so $v(xy)$ is indeed $\frac{2}{\sqrt{3}}(x^2-y^2)$

Question 3

0.1 Part a

Looking at diagram (a) we see two similar triangles which give rise to the relation

$$\frac{S_y}{S_x}=\frac{P_y}{P_x}$$

Using Cartesian coordinates

$$S_y=1-x_3, S_x=x_2$$

$$P_y=2, P_x=y_1$$

So by similar triangles

$$\frac{1-x_3}{x_2}=\frac{2}{y_1}$$

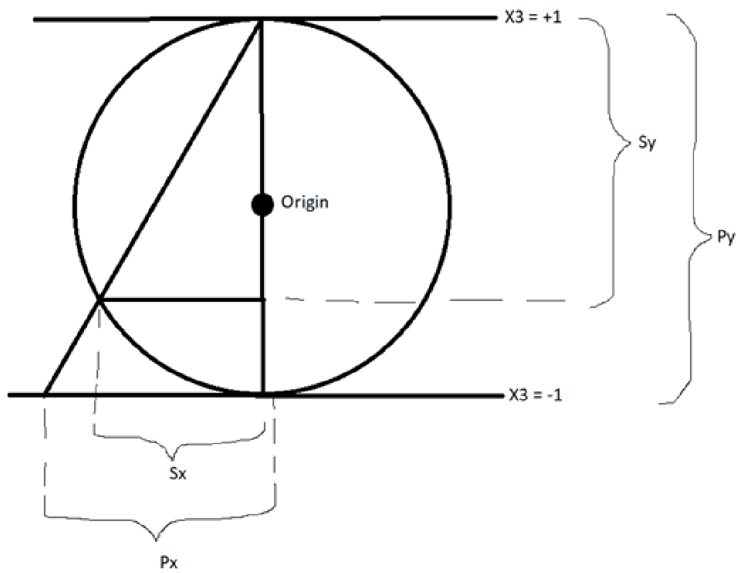

Figure 1: Diagram (a)

$$2x_2 = \frac{y_1}{1 - x_3}$$

$$y_1 = \frac{2x_2}{1 - x_3}$$

By rotational symmetry about the x_3 axis

$$y_2 = \frac{2x_1}{1 - x_3}$$

Providing

$$(y_1, y_2) = \left(\frac{2x_2}{1 - x_3}, \frac{2x_1}{1 - x_3} \right)$$

Part b

The answer is found in the same manner as part a except now x_3 moves above the lower (x_1, x_2) plane and so we have the mapping

$$1 - x_3 \to 1 + x_3$$

Which yields

$$(z_1, z_2) = \left(\frac{2x_2}{1+x_3}, \frac{2x_1}{1+x_3} \right)$$

Question 4

We wish to evaluate

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{1}{2} g^{\lambda'\rho'} \left(\partial_{\mu'} g_{\nu'\rho'} + \partial_{\nu'} g_{\rho'\mu'} - \partial_{\rho'} g_{\mu'\nu'} \right)$$

First evaluate the bracketed term

$$\partial_{\mu'} g_{\nu'\rho'} + \partial_{\nu'} g_{\rho'\mu'} - \partial_{\rho'} g_{\mu'\nu'}$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu'}} \frac{\partial x^\rho}{\partial x^{\rho'}} + g_{\nu\rho} \left(\frac{\partial x^\rho}{\partial x^\rho} \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}} + \frac{\partial x^\rho}{\partial x^{\nu'}} \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\rho'}} \right)$$

$$+ \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\nu g_{\rho\mu} + g_{\rho\mu} \left(\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^\rho}{\partial x^{\nu'} \partial x^{\rho'}} + \frac{\partial x^\mu}{\partial x^{\rho'}} \frac{\partial^2 x^\rho}{\partial x^{\nu'} \partial x^{\mu'}} \right)$$

$$- \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \partial_\rho g_{\mu\nu} - g_{\mu\nu} \left(\frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial^2 x^\mu}{\partial x^{\rho'} \partial x^{\nu'}} + \frac{\partial x^\nu}{\partial x^{\mu'}} \frac{\partial^2 x^\mu}{\partial x^{\rho'} \partial x^{\nu'}} \right)$$

Using the symmetry of g and exchanging indices

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\rho}{\partial x^{\rho'}} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) + g_{\nu\rho} (\frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}}$$

$$+ \frac{\partial x^\rho}{\partial x^{\nu'}} \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\rho'}} + \frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial^2 x^\nu}{\partial x^{\nu'} \partial x^{\rho'}} + \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial^2 x^\nu}{\partial x^{\nu'} \partial x^{\mu'}} - \frac{\partial x^\rho}{\partial x^{\nu'}} \frac{\partial^2 x^\nu}{\partial x^{\rho'} \partial x^{\mu'}} - \frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial^2 x^\nu}{\partial x^{\rho'} \partial x^{\nu'}}$$

Collecting like terms

$$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\rho}{\partial x^{\rho'}} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) + 2 g_{\nu\rho} \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\nu'}}$$

making the total expression for Γ

$$\frac{1}{2}(\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\frac{\partial x^{\rho'}}{\partial x^{\beta}}g^{\lambda\beta})(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\rho}}{\partial x^{\rho'}}(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\rho\mu}-\partial_\rho g_{\mu\nu})+2g_{\nu\rho}\frac{\partial x^{\rho}}{\partial x^{\rho'}}\frac{\partial^2 x^{\nu}}{\partial x^{\mu'}\partial x^{\nu'}})$$

$$=\frac{1}{2}g^{\lambda\beta}\delta^{\rho}_{\beta}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\rho\mu}-\partial_\rho g_{\mu\nu})+g_{\nu\rho}g^{\lambda\beta}\delta^{\rho}_{\beta}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\frac{\partial^2 x^{\nu}}{\partial x^{\mu'}\partial x^{\nu'}})$$

$$=\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\rho\mu}-\partial_\rho g_{\mu\nu})+\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\frac{\partial^2 x^{\lambda}}{\partial x^{\mu'}\partial x^{\nu'}})$$

Which is the transformation law obeyed by an affine connection as required.

Question 5

Part a

Perpendicularity is given by the condition

$$V^a W_a = 0$$

Contracting four velocity and acceleration yields

$$U^a\left(\frac{-1}{c^2}\partial_a\phi-\frac{1}{c^2}U_a\partial^b\phi U_b\right)$$

$$=\frac{-1}{c^2}U^a\partial_a\phi-\frac{1}{c^2}U^aU_a\partial^b\phi U_b$$

Since $U^aU_a=-1$

$$=\frac{-1}{c^2}U^a\partial_a\phi+\frac{1}{c^2}\partial^b\phi U_b$$

$$=0$$

So the vectors are perpendicular.

Part b

Removing the right hand side term

$$U^a A'_a = U^a \left(\frac{-1}{c^2} \partial_a \phi \right)$$

$$= \frac{-1}{c^2} U^a \partial_a \phi$$

For this to be zero we require

$$\frac{-1}{c^2} U^a \partial_a \phi = 0$$

$$U^0 \partial_0 \phi + U^i \partial_i \phi = 0$$

$$\frac{1}{c} \frac{\partial}{\partial t} \phi = -\frac{v^i}{c} \nabla^i \phi$$

which is satisfied in a number of cases. One being a time invariant potential $\frac{\partial}{\partial t}\phi = 0$ in which the particle always moves along an equipotential trajectory $v^i \nabla^i \phi = 0$.

Part c

We find a solution by considering the chain rule and using the provided definitions in the assignment sheet

$$\frac{dU^i}{ds} = \frac{1}{c} \frac{d(\gamma v^i)}{ds} = \frac{\gamma}{c} \frac{dv}{ds} + \frac{v^i}{c} \frac{d\gamma}{ds}$$

$$\frac{dU^i}{ds} = \frac{\gamma}{c} \frac{dt}{ds} \frac{dv^i}{dt} + \frac{v^i}{c} \frac{dt}{ds} \frac{d\gamma}{dt}$$

Because $c\frac{dt}{ds} = \gamma$ we have $\frac{dt}{ds} = \frac{\gamma}{c}$

$$\frac{dU^i}{ds} = \frac{\gamma^2}{c^2} \frac{dv}{dt} + \frac{v^i \gamma}{c^2} \frac{d\gamma}{dt}$$

We will need the quantity $\frac{d\gamma}{dt}$ which is found by considering the zero component of $\frac{dU^{mu}}{ds}$

$$\frac{dU^0}{ds} = \frac{dt}{ds}\frac{dU^0}{dt}$$

$$\frac{dU^0}{ds} = \frac{-1}{c^2}(\partial^0\phi + U^0\nabla_i U^i)$$

$$= \frac{-1}{c^2}(0 + \gamma GM\frac{\gamma v^i \hat{r}_i}{r^2 c})$$

$$= \frac{-\gamma}{c^2}GM\frac{\hat{r^i\gamma}v_i}{r^2 c}$$

$$= \frac{-\gamma^2}{c^3}\frac{GMv_i}{r^2}\hat{r^i}$$

Recalling that $\frac{dt}{ds} = \frac{\gamma}{c}$ we have

$$\frac{d\gamma}{dt} = -\frac{\gamma}{c^2}\frac{GM\hat{r}_i v^i}{r^2}$$

And so

$$\frac{dU^i}{ds} = \frac{\gamma^2}{c^2}\frac{dv^i}{dt} - \frac{v^i\gamma}{c^2}\frac{\gamma}{c^2}\frac{GM\hat{r}^i v^i}{r^2}$$

The vector v^i points in the direction of $\hat{r^i}$ so may be written $v\hat{r^i}$ giving

$$= \frac{\gamma^2}{c^2}\frac{dv^i}{dt} - \frac{\gamma^4}{c^4}\frac{GMv^2\hat{r^i}}{r^2}$$

Equating $\frac{dU^i}{ds}$ we need the quantity $\partial_i\phi$

$$\partial_i\phi = \frac{\partial}{\partial x_i}\left(\frac{-GM}{(x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}}\right)$$

$$= GM\frac{1}{r^3}r^i = GM\hat{r^i}\frac{1}{r^2}$$

Which gives

$$\frac{dU^i}{ds} = \frac{-1}{c^2}\left(\frac{GM\hat{r^i}}{r^2} + \frac{\gamma^2 GMv^2\hat{r^i}}{c^2 r^2}\right)$$

So

$$\frac{-1}{c^2}\left(\frac{GM\hat{r^i}}{r^2}+\frac{\gamma^2 GMv^2\hat{r^i}}{c^2r^2}\right)=\frac{\gamma^2}{c^2}\left(\frac{dv^i}{dt}-\frac{GMv^2\hat{r^i}}{c^2r^2}\right)$$

$$\frac{-1}{\gamma^2}\left(\frac{GM\hat{r^i}}{r^2}+\frac{\gamma^2 GMv^2\hat{r^i}}{c^2r^2}\right)=\frac{\gamma^2}{c^2}\left(\frac{dv^i}{dt}-\frac{GMv^2\hat{r^i}}{c^2r^2}\right)$$

$$\frac{-1}{\gamma^2}\left(\frac{GM\hat{r^i}}{r^2}+\frac{\gamma^2 GMv^2\hat{r^i}}{c^2r^2}\right)=\frac{dv^i}{dt}-\frac{GMv^2\hat{r^i}}{c^2r^2}$$

$$\frac{dv^i}{dt}=\frac{GM\hat{r^i}}{\gamma^2r^2}-\frac{GMv^2\hat{r^i}}{r^2c^2}+\frac{GMv^2\hat{r^i}}{r^2c^2}$$

$$=\frac{GM\hat{r^i}}{\gamma^2r^2}$$

$$=-\frac{GM}{r^2}\hat{r^i}\left(1-\frac{v^2}{c^2}\right)$$

As required

Question 6

Part a

Expanding the metric on the paraboloid

$$ds^2=Adu^2+Bdud\phi+Cd\phi^2$$

And evaluating the coefficients

$$A=\left(\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2\right)$$

$$=cos^2\phi+sin^2\phi+u^2$$

$$=1+u^2$$

$$B = \frac{\partial x}{\partial u}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial \phi} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial \phi}$$

$$= -cos\phi sin\phi u + sin\phi cos\phi u + 0$$

$$= 0$$

$$C = \left(\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2\right)$$

$$= u^2 sin^2\phi + u^2 cos^2\phi + 0$$

$$= u^2$$

Giving a metric

$$ds^2 = (1 + u^2)du^2 + u^2 d\phi^2$$

Part b

Christoffel Symbols are given by

$$\Gamma^a_{cb} = \frac{1}{2} g^{ad} \left(g_{bd,c} + g_{cd,b} + g_{cb,d}\right)$$

We know the quantity $g_{\mu\nu}$ obeys the relation

$$ds^2 = dx^\mu g_{\mu\nu} dx^\nu$$

From part (a) this gives us

$$dx^\mu g_{\mu\nu} dx^\nu = (1 + u^2)du^2 + u^2 d\phi^2$$

Meaning $g_{\mu\nu}$ is

$$g_{\mu\nu} = \begin{bmatrix} 1 + u^2 & 0 \\ 0 & u^2 \end{bmatrix}$$

The action of two g matrices on a quantity result in the original quantity and thus the value of the contravariant g must be such that, when multiplied by the covariant g, recovers the identity. This is satisfied by

$$g^{\mu\nu} = \begin{bmatrix} \frac{1}{1+u^2} & 0 \\ 0 & \frac{1}{u^2} \end{bmatrix}$$

We find our Christoffel symbols to be

$$\Gamma^1_{11} = \frac{1}{2} g^{11} \left(\frac{\partial}{\partial u} g_{11} + \frac{\partial}{\partial u} g_{11} - \frac{\partial}{\partial u} g_{11} \right)$$

$$= \frac{1}{2} \frac{1}{1+u^2} (2u) = \frac{u}{1+u^2}$$

$$\Gamma^1_{12} = \frac{1}{2} g^{11} \left(\frac{\partial}{\partial \phi} g_{11} + \frac{\partial}{\partial u} g_{21} - \frac{\partial}{\partial u} g_{21} \right)$$

$$= \frac{1}{2} \frac{1}{1+u^2} (0+0+0) = 0$$

$$\Gamma^1_{21} = \frac{1}{2} g^{11} \left(\frac{\partial}{\partial u} g_{21} + \frac{\partial}{\partial \phi} g_{11} - \frac{\partial}{\partial u} g_{12} \right)$$

$$= \frac{1}{2} \frac{1}{1+u^2} (0+0+0) = 0$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11} \left(\frac{\partial}{\partial \phi} g_{21} + \frac{\partial}{\partial \phi} g_{21} - \frac{\partial}{\partial u} g_{22} \right)$$

$$= \frac{1}{2} \frac{1}{1+u^2} (-2u) = \frac{-u}{1+u^2}$$

$$\Gamma^2_{11} = \frac{1}{2} g^{22} \left(\frac{\partial}{\partial u} g_{12} + \frac{\partial}{\partial u} g_{12} - \frac{\partial}{\partial \phi} g_{11} \right)$$

$$= \frac{1}{2} \frac{1}{u^2} (0+0+0) = 0$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} \left(\frac{\partial}{\partial \phi} g_{12} + \frac{\partial}{\partial u} g_{22} - \frac{\partial}{\partial \phi} g_{21} \right)$$

$$= \frac{1}{2}\frac{1}{u^2}(0 + 2u + 0) = \frac{1}{u}$$

$$\Gamma^2_{21} = \frac{1}{2}g^{22}\left(\frac{\partial}{\partial u}g_{22} + \frac{\partial}{\partial \phi}g_{12} - \frac{\partial}{\partial \phi}g_{12}\right)$$

$$= \frac{1}{2}\frac{1}{u^2}(2u + 0 + 0) = \frac{1}{u}$$

$$\Gamma^2_{22} = \frac{1}{2}g^{22}\left(\frac{\partial}{\partial \phi}g_{22} + \frac{\partial}{\partial \phi}g_{22} - \frac{\partial}{\partial \phi}g_{22}\right)$$

$$= \frac{1}{2}\frac{1}{u^2}(0 + 0 + 0) = 0$$

Giving the non-zero symbols to be

$$\Gamma^1_{11} = \frac{u}{1+u^2}, \Gamma^1_{22} = \frac{-u}{1+u^2}, \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{u}$$

Part c

Writing the condition for parallel transport in explicit components we have two equations

$$U^1\nabla_1 V^1 + U^2\nabla_2 V^1 = 0$$

$$\frac{\partial u}{\partial t}\left(\frac{\partial}{\partial u}V^1 - \Gamma^1_{11}V^1 - \Gamma^2_{11}V^2\right) + \frac{\partial \phi}{\partial t}\left(\frac{\partial}{\partial \phi}V^1 - \Gamma^1_{12}V^1 - \Gamma^2_{12}V^2\right) = 0$$

If $t = \phi$ then $\frac{\partial u}{\partial t} = 0$ and $\frac{\partial \phi}{\partial t} = 0$ leaving

$$\frac{\partial}{\partial \phi}V^1 - \Gamma^1_{12}V^1 - \Gamma^2_{12}V^2 = 0$$

We know the symbols from part (b), and are further given the condition that $u = u_0$ where u_0 is a positive constant thus

$$\frac{\partial}{\partial \phi}V^1 = \frac{1}{u_0}V^2 = 0$$

Second equation

$$U^1\nabla_1 V^2 + U^2\nabla_2 V^2 = 0$$

$$\frac{\partial}{\partial\phi}V^2 - \Gamma^1_{22}V^1 - \Gamma^2_{22}V^2 = 0$$

Using our known Γ values

$$\frac{\partial}{\partial\phi}V^2 = \Gamma^1_{22}V^1$$

$$= \frac{-u_0}{1+u_0^2}V^1$$

This gives the coupled set of differential equations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{u_0} \\ \frac{-u_0}{1+u_0^2} & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Where we have made the substitutions $\frac{\partial V^1}{\partial\phi} = x'$, $\frac{\partial V^2}{\partial\phi} = y'$ and $V^1 = x$, $V^2 = y$. This system has the standard trigonometric solution

$$\begin{bmatrix} V^1 \\ V^2 \end{bmatrix} = \begin{bmatrix} c_1\sqrt{\frac{1}{u_0^2+1}}sin(\sqrt{\frac{1}{1+u_0^2}}\phi) - c_2\sqrt{\frac{1}{u_0^2+1}}cos(\sqrt{\frac{1}{1+u_0^2}}\phi) \\ c_1cos(\sqrt{\frac{1}{1+u_0^2}}\phi) + c_2sin(\sqrt{\frac{1}{1+u_0^2}}\phi) \end{bmatrix}$$

Imposing the initial conditions $V^1 = 1$ and $V^2 = 0$ we return

$$\begin{bmatrix} V^1 \\ V^2 \end{bmatrix} = \begin{bmatrix} c_2\sqrt{\frac{1}{u_0^2+1}} \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus

$$c_2 = \sqrt{u_0^2+1}$$

$$c_1 = 0$$

Giving particular solutions

$$\begin{bmatrix} V^1 \\ V^2 \end{bmatrix} = \begin{bmatrix} -cos(\sqrt{\frac{1}{1+u_0^2}}\phi) \\ \sqrt{u_0^2+1}sin(\sqrt{\frac{-1}{1+u_0^2}}\phi) \end{bmatrix}$$

Appendix

Following is the MATLAB software used in the generation of the rotation matrices for problem 2.

```
syms T;

R = [ cos(T) , sin(T) ;
      -sin(T), cos(T) ];

EQN = [ 2 , 1 ;
        1 , 2 ];

f = R.' * EQN * R;

simplify(f)
```

```
ans =

[ 2 - sin(2*T),     cos(2*T)]
[     cos(2*T), sin(2*T) + 2]
```

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