On the Saha Equation and the Music of the Sun

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Saha Equation

Emission and absorption spectral lines are an integral part of human observations of the cosmos and as such understanding their physical origin provides great insight into the environment around us. Use of the Saha equation provides this pinnacle insight and understanding the implications which arise from it provide a means of understanding the conditions which create the ubiquitous emission and absorption spectral lines.

Spectral lines arise from the excitation of electrons in atomic orbitals and due to this a limit on the strength of spectral lines is imposed by the number of atoms ionised such that their electrons are no longer able to participate in spectral activity. The ratio of ionised atoms in a gas to neutral atoms is dependent upon a number of quantitative factors summarised by the Saha equation:

$$\frac{N_{i+1}}{N_i} = A\frac{(kT)^{1.5}}{N_e}\frac{g_{i+1}}{g_i}e^{-x/kT}$$

A derivation beginning from the Boltzmann relation can be reached through the reference section (Princeton 2018). Here the ratio of atoms in a particular ionisation state (N_i) to that in another ionisation state (N_{i+1}) is related to the quantum mechanical weights of the two ionisation states (g_i and g_{i+1}), the temperature of the gas (T) the ionisation potential (x), the number density of free electrons (N_e), Boltzmann's constant (k) and a constant (A). Using this equation the number of atoms available to contribute to spectral line strength compared to the total number of atoms can be quantified. To illustrate this point the simple case of Hydrogen gas is presented. A plot showing the number of ionised Hydrogen atoms (N_{II}) in a gas as a proportion of total Hydrogen atoms (N_{total}) as a function of temperature is shown in figure 1.

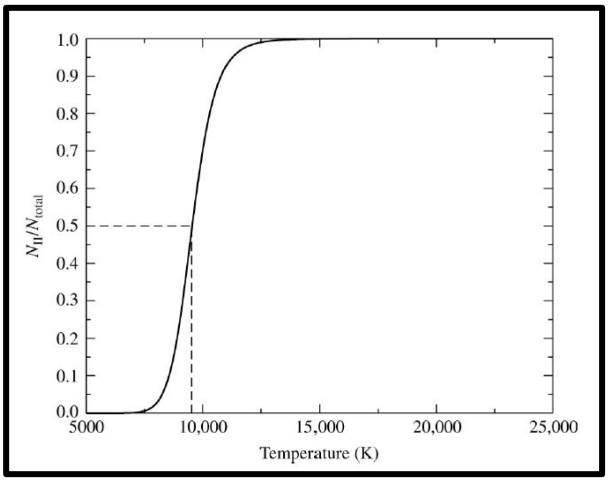

Figure 1: Proportion of ionised Hydrogen atoms present (N_{II}) in Hydrogen gas as a function of temperature (Rowell 2018)

To gain bearing on the number of atoms available to participate in spectral activity we observe it to be given by the complement of the above plot by the following mathematics:

$$\frac{N_I}{N_{Total}} = \frac{N_{total} - N_{II}}{N_{total}} = 1 - \frac{N_{II}}{N_{total}}$$

And so we expect maximum neutral Hydrogen density and hence spectral line strength for low temperatures with a sharp decay toward higher temperature. In the displayed example of Hydrogen (Figure 1) we note the 50% mark is reached for temperatures just shy of 10,000K.

The second factor affecting spectral strength is inter-particle interactions causing excitation of electrons into higher orbitals, affecting their ability to then emit and absorb photons. Excited (N_B) to relaxed (N_A) atomic state dependence on temperature is summarised by the Boltzmann equation:

$$\frac{N_B}{N_A} = \frac{g_B}{g_A} exp\left[\frac{-(E_B - E_A)}{kT}\right]$$

Where quantum mechanical weight of each atomic state is again represented by g_B for the relaxed state and g_A for the excited state (Rowell 2018). The energy of each atomic state is represented by E_A and E_B for relaxed and excited states respectively. From this we see the number of excited atoms and hence spectral strength now increases with temperature, due to the fractional negative exponential. Taking the simple case of Hydrogen for demonstrative purposes we can produce the graph seen in figure 2.

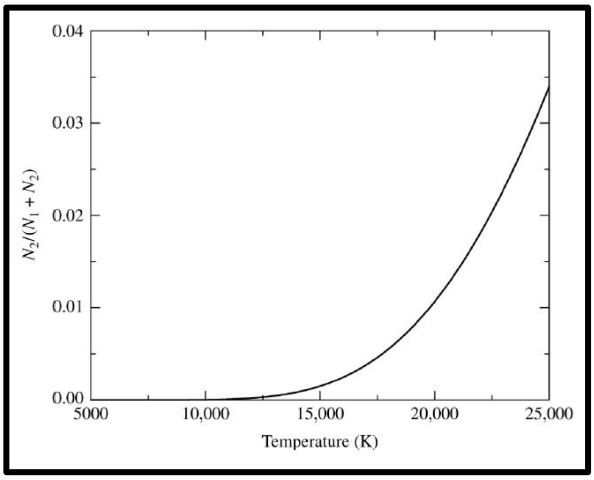

Figure 2: Ratio of excited (N_2) Hydrogen atoms of the total un-ionised atom number as a function of temperature (Rowell 2018)

Here we see the spectral strength increase exponentially as expected by Boltzmann theory. To gain the ultimate insight into spectral behaviour the union of both the Boltzmann equation, which provides the number of excited atoms, and the Saha equation, which provides an upper limit on the atoms available to partake in spectral activity, are considered. For Hydrogen the combination of both effects can be seen in figure 3, which combines the effects of figure 2 and the compliment of figure 1.

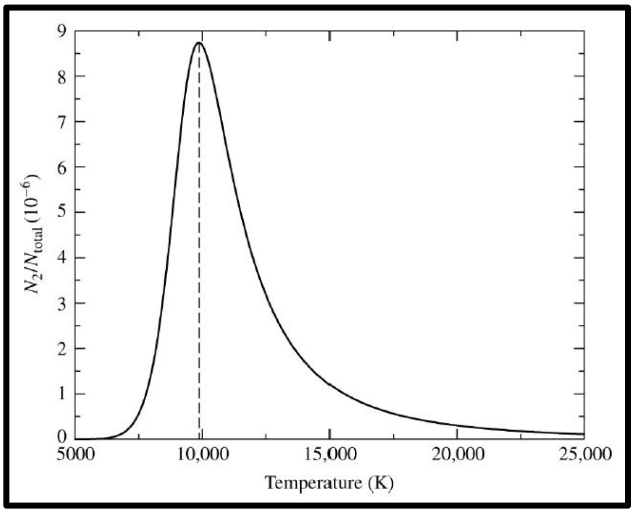

Figure 3: Proportion of excited Hydrogen atoms present (N_2) in Hydrogen gas as a function of temperature (Rowell 2018)

We note a maximum peak around the 10,000k mark for Hydrogen at which point the number of atoms available to emit and absorb photons is maximised. By observing the Saha and Boltzmann equation it can be gleamed that this peaked curve is not specific to Hydrogen (Rowell 2018) but is instead a property arising from the balance between thermal excitation and ionization in any gas.

Understanding such the Saha equation provides the upper limit to spectral strength and provides the physical connection between spectral line strength and ionization of gas particles. The insight it provides is well spread and is applicable to all gases, making it a useful tool in the analysis of the astrophysical space that surrounds us.

Music of the Sun

Life on earth would not be possible without the existence of the sun and as a consequence of this understanding the suns existence can provide major insights into how stars in general can form and their respective structures. A key component of this understanding stems from Helioseismology where solar surface waves can be analysed to provide an insight into the internal structure of the sun.

Following an earthquake oscillations continue to propagate through the earth's surface for some time afterwards (Rowell 2018) however due to the high density of earth's interior dissipative losses quickly cause the decay of oscillatory behaviour. These losses are not as prominent within the sun however due to its gaseous nature and oscillations are observed (Thompson 2004) to occur in seeming perpetuity as a result of this. Such oscillations are mediated by pressure waves (p waves) where the oscillation restoring force is supplied by the pressure difference created by the wave and gravity wave (g waves) where the restoring force is provided by the medium buoyancy and hence gravity. G waves which propagate on the surface of a body are referred to as f waves (check) (Rowell 2018). In order to construct standing waves within the solar body the centre point of the sun must, by necessity, be a point of no displacement for p and g waves. However, the surface is free to oscillate and as such the propagation of these waves in the suns interior can be detected by us on the solar surface. A common detection method utilises Michelson Dopler Interferometry where dopler shift caused by solar rotation is used to find spatial positioning of the surface waves (Rowell 2018).

A theoretical model can be constructed in an attempt to match what is observed on the solar surface and used to determine the internal structure. Standing waves in one dimension can be modelled via the use of a Fourier series. The standard equation for generic periodic wave structure is represented by:

$$f(x) = \mathrm{C} + \sum_{n=1}^{N} \left(A_n \sin\left(\frac{2n\pi x}{T} + \phi_n \right) \right)$$

C represents some constant offset, T the period of the oscillation and ϕ_n a phase offset (Rowell 2018). Once a system has been modelled to a sufficient standard by a particular sum the weighting of the constituent frequencies, represented by the constants A_n, can be observed and used to find what frequencies are present in the system. This process is known as Fourier analysis and the same process can be extended to model spherical harmonics in a 3-D sphere, such as the surface of our sun. The mathematical extension, known as a multi-pole expansion, is as follows:

$$A(\theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{-l}^{l} a_{lm}(t) P_l^{|m|}(\theta) e^{im\phi}$$

Where $P_l^{|m|}(\theta)$ are the orthogonal Legendre Polynomials as a function of latitude angle θ, and $e^{im\phi}$ encapsulates the periodic nature of the wave harmonics as a function of azimuthal angle ϕ (Wolfram 2018). The constants in this sum can then be adjusted until the resulting plot resembles the surfaces waves observed on the sun. Once achieved the relative weights of the coefficients a_{lm} can be analysed to determine which frequencies are present within the body. Results of this process are represented in figure 4, showing the extent at which frequencies are present and their individual

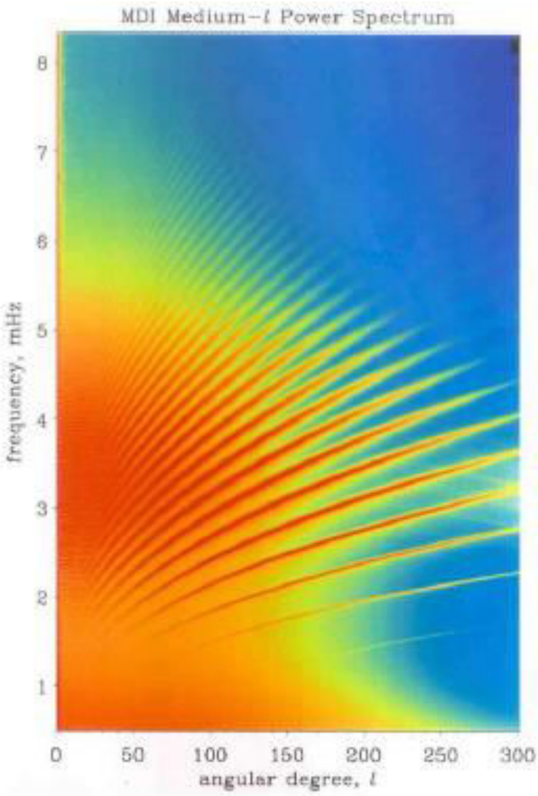

Figure 4: Oscillatory frequencies present upon the solar surface at differing angular locations (Rowell 2018)

angular position.

Mathematical modelling combined with solar observations provides numerical answers to the frequency and wavelength of waves present within the sun. Wave speed (v) can then be calculated by the product of both wavelength and frequency (since $v = f\lambda$). Bulk modulus (B) can be calculated through further mathematical modelling (Rowell 2018) and then via the relation:

$$v = \sqrt{\frac{B}{\rho}}$$

Solar density is able to be inferred.

Understanding the waves present within the sun provides crucial insight into the internal structure and of particular importance the density of the sun which is hard calculated otherwise. The understanding of solar structure greatly aids the knowledge of how the most crucial life source, the sun itself, exists and gives insight into the conditions necessary for such life-sustaining bodies within the universe.

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