Measurement of the local gravitational strength via the use of Kater's Pendulum

Joseph Lyle Pritchard

University of Adelaide, Adelaide, Australia

Introduction

Pendulums have been a source for the local determination of gravity for many years and can be used in a variety of ways to determine a value for the local gravitational strength. Simple pendulums are a useful tool in measuring gravity however due to the difficulty in constructing a true simple pendulum there are limits imposed on the accuracy of this method. An alternative is to use Kater's Pendulum which consists of two masses of different weight on either end of a pendulum shaft. Starting from the equation for the swing period of a simple pendulum it can be shown Kater's pendulum obeys by the equation:

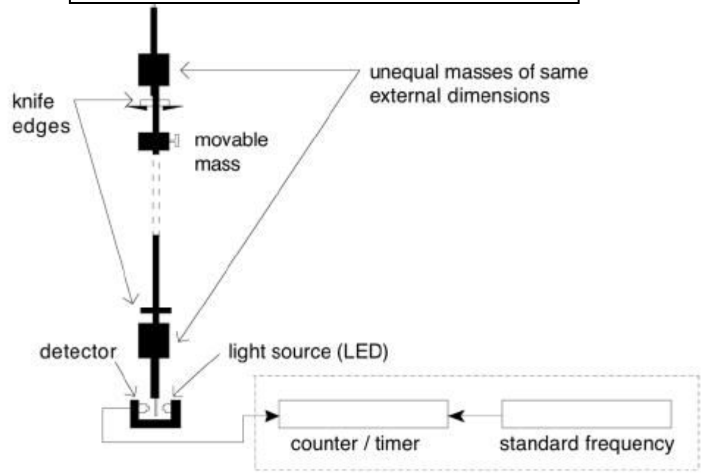
$$\frac{4\pi^2}{g} = \frac{T_1^2 + T_2^2}{2(h_1 + h_2)} + \frac{T_1^2 - T_2^2}{2(h_1 - h_2)} \tag{1}$$

Where T_1 and T_2 are the pendulum swing periods from either end, h_1 and h_2 are the distances from either end to the centre of mass and g is the local gravity strength. A reference featuring the derivation of this equation can be found in the reference section (Field & Hazlett 2001). All values can be measured with considerable accuracy bar for (h_1-h_2). This uncertainty can however be minimised by ensuring T_1 and T_2 are approximately equal, making the second fraction small compared with the first. A consequence of this accuracy is an ability to calculate gravity with greater accuracy than would be attained via the use of a simple pendulum. It is this fact that motivates this experiment and will be used to attain an accurate value of gravity.

Optimisation of Apparatus

Optimum apparatus setup was crucial in ensuring the gathered results were as accurate as possible. The pendulum arm utilised in the experiment featured two unequal masses (labelled m_1 and m_2) of equal external dimensions fixed to opposite ends of a steel bar. Either end of the pendulum featured knife edges, allowing for accurate suspension. An additional, adjustable mass was present between the two end masses to allow adjustment of the systems centre of mass. A light gate was connected to a computer such that the timing of repeated light beam obstructions by the swinging pendulum could be recorded over a given time interval. The pendulum arm was then suspended such that when the pendulum was hanging freely one pendulum end was centred in the light gate, ensuring the light gate and pendulum were correctly aligned. A diagrammatic representation of this setup can be seen in figure 1.

Figure 1: Kater's Pendulum Setup



Optimising amplitude of oscillations

An optimum amplitude for the experiment was one which obeyed the required relation that period be independent of amplitude, as can be observed from Kater's equation (1) where period is defined in terms of h_1 , h_1 , g and other constants all of which are independent of amplitude. Amplitude was then chosen such that experimental uncertainty was minimised.

The pendulum was set to swing from varying starting amplitudes to determine upper and lower bounds for which it was appropriate to assume period and amplitude independence. For 500 pendulum swings the period and amplitude were recorded to confirm the relationship between the two and provide a region for which the assumption was accurate. This process was completed for higher amplitudes (12cm) and lower amplitudes (3cm) to establish the relationship.

The most appropriate starting amplitude was then determined exactly by setting the pendulum in motion for amplitudes within the measured bounds calculated prior. Ten cycles were recorded per amplitude and plotted against period using software which also displayed the standard error present in each set of period measurements. The region for which amplitude was very nearly independent of period was then found. Period uncertainty was used as a guide to then choose the most optimum amplitude within this region.

Optimum weight position

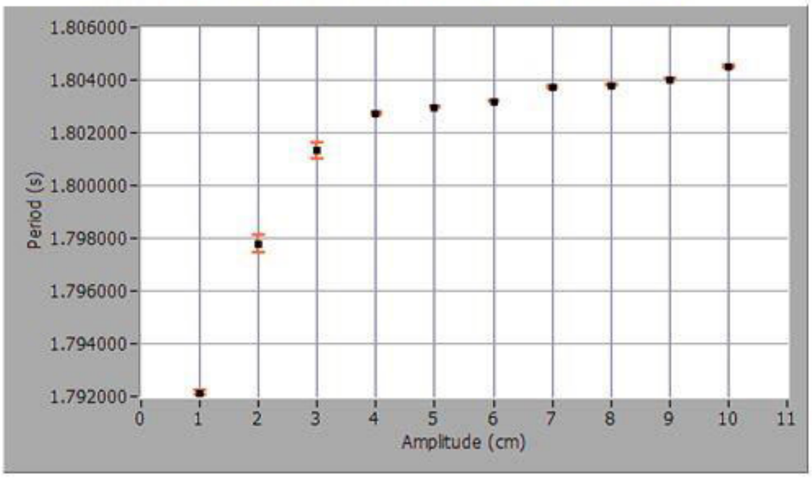
Mass one (m_1) was used to support the pendulum such that mass two (m_2) hung freely. The central adjustable mass was positioned such that it aligned with the lowest horizontal position marker along the pendulums shaft. This marking was labelled as position 1 with the second lowest marking labelled as position 2 and so on up to position 11. Swing period of one pendulum cycle was then recorded for each weight position and stored in the system software. This process was repeated with mass 2 resting on the supports, taking care that pendulum markings were labelled to be consistent with those previous (the lowest marking was now labelled position 11 and so on). These results were then plotted on a graph to determine the relationship. The weight position for which the swing periods were most closely matched was the optimum adjustable weight positioning to minimise the experimental dependence on the value (h_1-h_2), which has a large uncertainty. The reason for this is clarified in the determination of experimental values.

Results for optimum apparatus set-up

From the initial measurements taken it was observed amplitudes of 5cm and below meet the requirement that period of oscillation be independent of amplitude. Through further analysis it was found that the optimum starting amplitude lied within the bounds of 4 and 6cm (Graph 1). After many oscillations in the initial trial, when the amplitude dropped below 3cm, the recording software automatically amplified the display for the period. This made the displayed results hard to interpret and so amplitudes below this level were not suitable (Results showing this feature can be viewed in appendix section A). In making precise measurements of period and amplitude dependence it was seen that below the 4cm level amplitude begins to rapidly decay (Graph 1). The cause was believed to be friction between the knife edges and supports, which provided a lower bound for the optimum amplitude to fit with the requirements.

For higher amplitudes of 5cm up to 12cm there was an observed relation between amplitude and period, seen by a steady rise in the graph (Graph 1). This result provided an upper bound of 6cm on the amplitude options as above this amplitude-period dependence was increased. Further results supporting this conclusion are included in appendix section B.

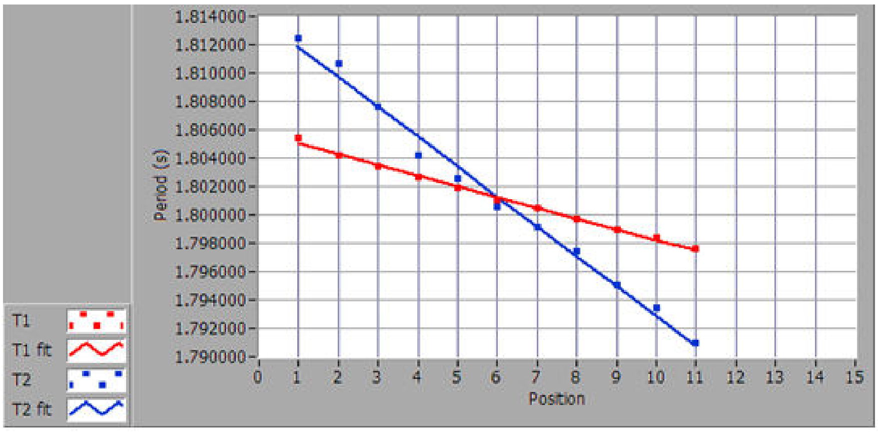
Graph 1: Starting Amplitudes against Pendulum Period Trial 2



Within the necessary bounds of 4-6cm the starting amplitude of 4cm was selected as best since it had a low uncertainty, represented by narrow error bars, and lied most securely in the region of amplitude-period independence.

Suitably choosing a weight position to minimise the value of $T_1^2 - T_2^2$ resulted in position 6 being chosen which was approximately half way along the pendulum shaft. This was seen as the point where the graphs of period one and period two against position intersect (Graph 2).

Graph 2: Periods 1 and 2 against Adjustable Weight Position



Discussion of results for apparatus set-up

Based on empirical evidence it was concluded a starting amplitude of 4cm and adjustable weight positioning of 6 most suitably minimised the uncertainty in measurement and matched with the requirements of the experiment.

Below the 4cm amplitude mark a strong relationship began to take hold causing dramatic decreases in period. It was believed this effect was due to friction within the system, potentially between the supporting knife edges and the pendulum's support. Low amplitudes were also met with an automated scaling in software display which led to hard-to-interpret results. This is an issue in the software and by choosing higher starting amplitudes this potential error was avoided.

Above this amplitude the data also began to climb at an increasing rate as period and amplitude became dependent on one another. This made higher starting amplitudes less suitable for the needs of this experiment.

In an effort to minimise the uncertainty of the final value of g, the (h_1-h_2) dependence was minimised by setting T_1 approximately equal to T_2. From the various adjustable weight positions trialled this occurred for position marking 6. Taking this precaution greatly increased the accuracy of the final result by limiting the effect of the hard to measure (h_1-h_2) on the final accuracy. It was noted that more closely matching periods could be achieved if half integer position markings were used however considering the almost exact cross-over seen in the results an improvement in this regard was likely to be outweighed by uncertainties in other variables.

Determination of Experimental Values and Gravity Strength

Calculation of pendulum midpoint

Through the use of torque arguments it can be shown that:

$$\frac{h_2}{h_1} = \frac{s_1}{s_2}$$

Where h_1 and h_2 are the distances to the centre of mass from mass 1 and mass 2 respectively and S_1 and S_2 correspond to the scale readings of mass 1 and mass 2 when placed on weighing scales (A derivation of this fact is included in appendix section C). Values for h_1 and h_2 were then calculated via the use of a travelling microscope. The pendulum was positioned on the travelling microscope and the microscope crosshairs were focused to minimise the presence of parallax error. Values for the positions of the knife edges on either end of the pendulum were then recorded. To ensure readings were accurately taken the measurements were recorded with both eyes open to reduce eye strain. The difference of the two readings then provided the length between the knife edges and the value for (h_1 + h_2).

Now having the value of (h_1 + h_2) and the ratio of h_1 to h_2 it was possible to determine their individual values by direct substitution and thus the distance (h_1 - h_2).

Determination of swing period

Periods T_1 and T_2 were determined by first orienting the pendulum such that it was supported by mass two. Period values were then recorded for 10 sets of 10 oscillations and plotted to allow for the determination of period 1 (T_1). Repeating the process with mass one at the support allowed for period values of period 2 (T_2) to be calculated and plotted. The results were then analysed to reach a value for the periods and their respective mean standard errors.

Results for experimental values of midpoint and swing period

Placing the masses on the scales and taking the scale readings to find the ratio of h_1/h_2 discerned:

$$\frac{h_2}{h_1} = \frac{s_1}{s_2} = \frac{1614.5g}{618.3g} = 2.6110 \pm 5 \times 10^{-4}$$

With scale reading uncertainty of ±0.1g being the smallest reading presented by the scales. Derivation of the ratio uncertainty is included in appendix section D.

When placed upon the travelling microscope the distance position of knife-edge 1 was measured to be:

$$K1 \ = \ 90.8950\,cm \ \pm \ 0.05\,mm$$

For knife edge 2 the distance was measured as:

$$K2 \ = \ 10.0635\,cm \ \pm \ 0.05\,mm$$

Where the uncertainty was taken as half the smallest travelling microscope increment. The value for (h_1 + h_2) was then given by the difference:

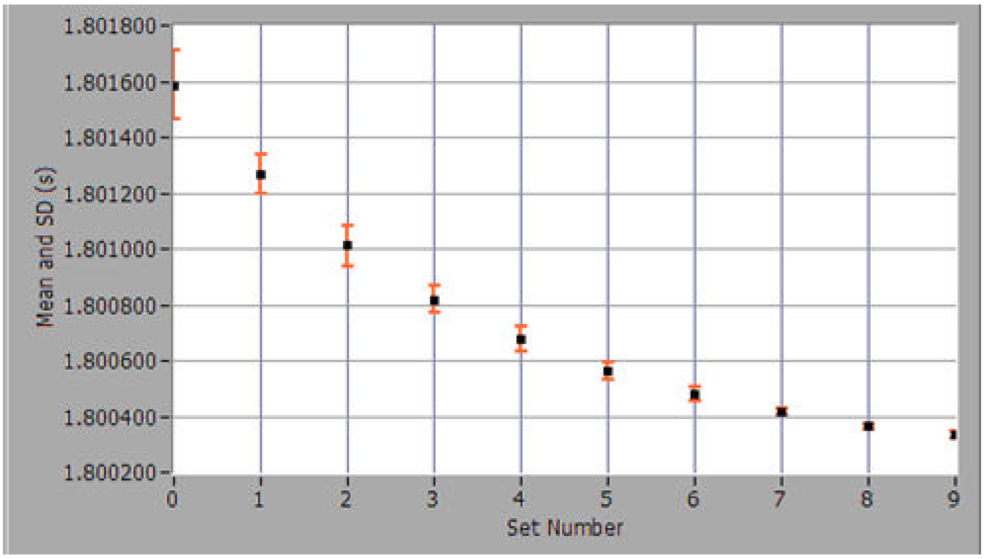
$$(h1 \ + \ h2) \ = \ K1 - K2 \ = \ 80.8315cm \ \pm 0.007cm$$

Calculating the difference of h_1 and h_2 was done by utilising the ratio relationship and their sum. A full derivation of the result and uncertainty can be found in appendix sections E and D respectively. The value for (h_1 - h_2) was found to be:

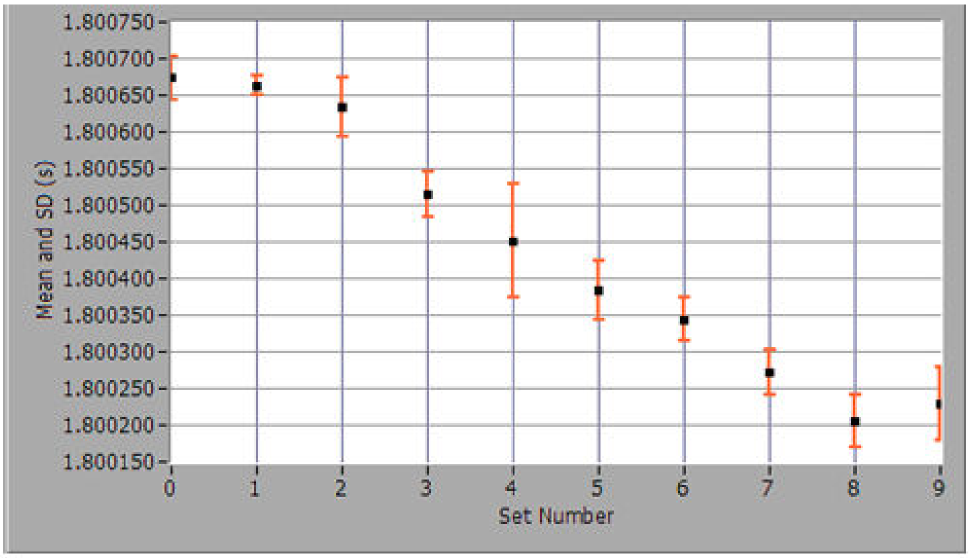
$$h_1 - h_2 = -36.0619cm \pm 0.02cm$$

In using system software to record the values of swing periods T_1 and T_2 for ten sets of ten cycles each the mean period along with its standard error was calculated using the software. The resulting graphs are shown below (Graphs 3 and 4).

Graph 3: Ten Sets of Ten Swings for T_1



Graph 3: Ten Sets of Ten Swings for T_2



Software analysis on these measurements resulted in mean values for T_1 and T_2 as follows:

$$T_1 = 1.80075 \pm 4 \times 10^{-5} s$$

$$T_2 = 1.80044 \pm 2 \times 10^{-5} s$$

Where uncertainty was given by the standard error of the mean provided in software analysis.

By substituting the measured values into Kater's equation (1) a value for gravity was determined. The derivation of the uncertainty is included in appendix section D. The value for gravity is then:

$$g = 9.787 \pm 0.002 ms^{-2}$$

Discussion of results for experimental values of midpoint, swing period and gravity

In calculating values dependant on the lengths h_1 and h_2 it was noted that varying levels of accuracy were present. For the sum ($h_1 + h_2$) the uncertainty was 7 x 10^{-3}cm whereas the difference ($h_1 - h_2$) had uncertainty 2 x 10^{-2}cm. As periods T_1 and T_2 had been set to be approximately equal it was expected the increased uncertainty in the difference, while large, would not have a significant impact on the accuracy of the final value for g.

The swing periods T_1 and T_2 were recorded to have an uncertainty of ± 4 x 10^{-5} and ± 2 x 10^{-5} respectively. Due to the higher level of accuracy compared to the other measurements it was unlikely the values for period would provide a significant contribution to the overall error in g.

A decay in the swing period was noticeable over time however as this decay occurred in the 5th and 6th decimal place it neglected to provide a significant uncertainty leading to accurate results. A possible cause for this was friction present between the knife edges and the supporting apparatus, which was believed to have also caused a rapid decay in period for small starting amplitudes when optimising the apparatus.

The value for g was measured to within 4 decimal places of accuracy which in consultation with literature (Hill 2018) matches the limits of accuracy expected in the use of Kater's pendulum and so this result is expected.

Comparing the calculated value of g to the literature value of:

$$g = 9.7972\, ms^{-2}$$

It was noted the result achieved was only accurate to the 2nd decimal place, with rounding. A liable cause for this was the friction believed to have been present between the knife edges and the pendulum supports, which caused a decline in period for small swing amplitudes. In avoiding small starting amplitudes the effect of this was minimised however its presence may have been the cause for the reduced precision of the result. In future experiments care should be taken to ensure no such friction is present.

Conclusion

The value for g calculated was:

$$g = 9.7873 \pm 0.002 ms^{-2}$$

This value fails to capture the cited literature value within experimental error, despite being accurate to within 2 decimal places. This is likely due to erroneous measurements caused by friction and other systematic errors.

Reference list

Hill, G 2018, 'Kater's pendulum', practical notes distributed in the topic 2510 Physics IIA, University of Adelaide, viewed 29 May 2018, <https://myuni.adelaide.edu.au/courses/33284/files/2469010?module_item_id=1128399>

Hill, G 2018, 'Practical notes', practical notes distributed in the topic 2510 Physics IIA, University of Adelaide, viewed 29 May 2018, < https://myuni.adelaide.edu.au/courses/33284/pages/practical-notes-2018?module_item_id=1128397>

Field, S & Hazlett, E 2001, *Kater's pendulum*, CSU, viewed 1 June 2018, <https://physics.csuchico.edu/ayars/427/handouts/Kater_Field-Hazlett.pdf>.

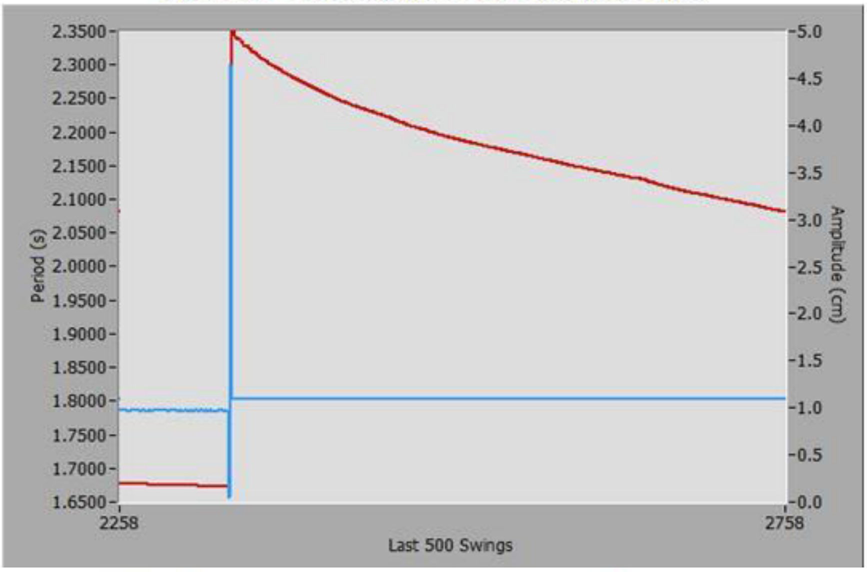
Hill, G 2018, 'Experimental value for gravity', resource provided in Physics IIA practical laboratories, University of Adelaide, on 4 June 2018

Appendices

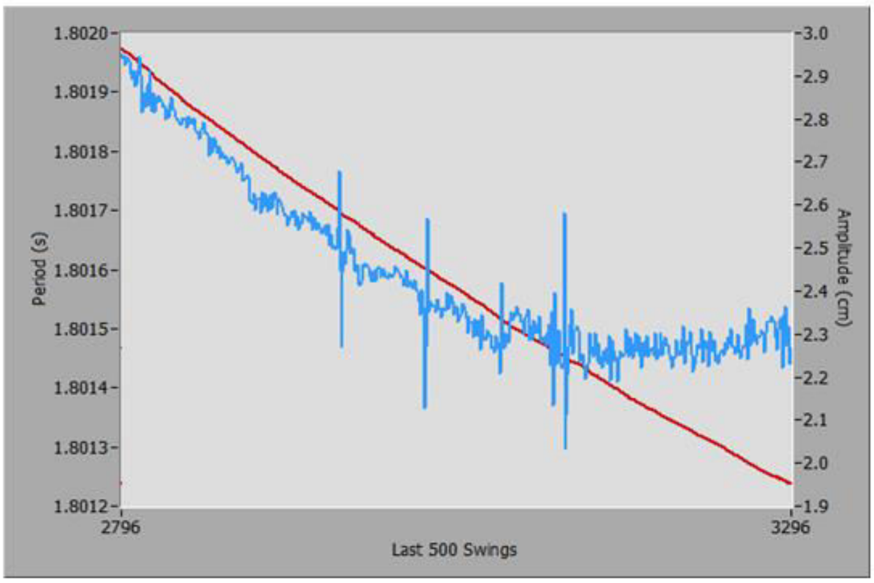
Section A – Initial trial for Period and Amplitude Independence

Blue line represents period of oscillations and red line represents amplitude of oscillations.

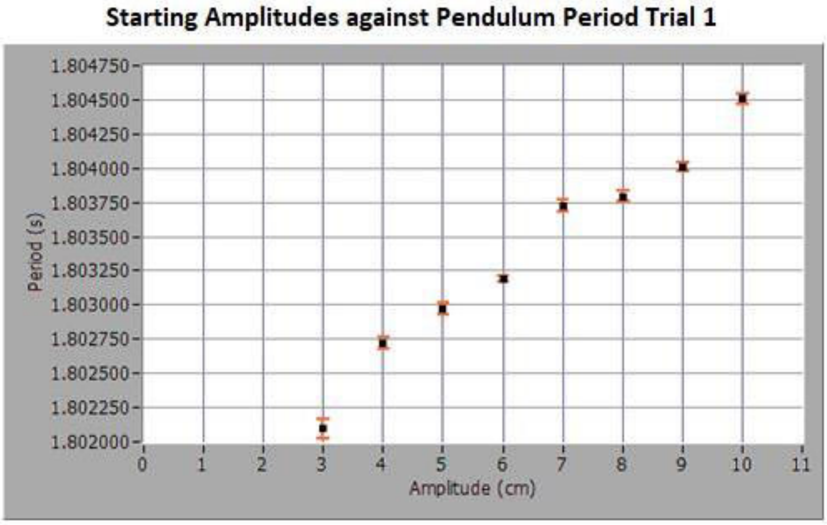
Effect of 3-5cm Amplitudes on Pendulum Period



Effect of 3cm Amplitudes and Below on Pendulum Period



Section B – Relation between Amplitude and Period with Errors



Section C – Derivation of Scale Reading and Rod Length Relation

Using torque arguments, clockwise torques must equal anticlockwise torques:

$$\sum \tau_C = \sum \tau_{AC}$$

$$F_1 h_1 = F_2 h_2$$

$$\frac{F_1}{F_2} = \frac{h_2}{h_1}$$

F_1 and F_2 given by scale readings s_1 and s_2 so:

$$\frac{s_1}{s_2} = \frac{h_2}{h_1}$$

Section D – Error derivation

Uncertainty in gravity is given by:

$$\sigma_g{}^2 = \sigma_{T_1}{}^2 \left(\frac{\partial g}{\partial T_1}\right)^2 + \sigma_{T_2}{}^2 \left(\frac{\partial g}{\partial T_2}\right)^2 + \sigma_{h_1}{}^2 \left(\frac{\partial g}{\partial h_1}\right)^2 + \sigma_{h_2}{}^2 \left(\frac{\partial g}{\partial h_2}\right)^2$$

Calculating uncertainties in each variable, using uncertainties attained upon measurement:

$$\sigma_{T_1}{}^2 = 1.6 \times 10^{-9} s$$

$$\sigma_{T_2}{}^2 = 4 \times 10^{-10} s$$

$$\sigma_{\frac{h_2}{h_1}}{}^2 = \sigma_{s_2}{}^2 \left(\frac{\partial(\frac{h_2}{h_1})}{\partial s_2}\right)^2 + \sigma_{s_1}{}^2 \left(\frac{\partial(\frac{h_2}{h_1})}{\partial s_1}\right)^2 = \sigma_{s_2}{}^2 \left(-\left(\frac{s_1}{s_2{}^2}\right)\right)^2 + \sigma_{s_1}{}^2 \left(\frac{1}{s_2}\right)^2$$

$$\sigma_{\frac{h_2}{h_1}}{}^2 = (10^{-4})^2\left(-\left(\frac{1.6145}{0.6183^2}\right)\right)^2 + (10^{-4})^2\left(\frac{1}{0.6183}\right)^2 = 2.79413 \times 10^{-7}$$

$$\sigma_{h_1+h_2}{}^2 = \sigma_{k_1}{}^2 + \sigma_{k_2}{}^2 = 2 \cdot (5 \times 10^{-5})^2 = 5 \times 10^{-9} m^2$$

$$\sigma_{h_1}{}^2 = \sigma_{h_1+h_2}{}^2\left(\frac{\partial h_1}{\partial(h_1+h_2)}\right)^2 + \sigma_{\frac{h_2}{h_1}}{}^2\left(\frac{\partial h_1}{\partial(\frac{h_2}{h_1})}\right)^2$$

In appendix section E it is shown:

$$h_1 = \frac{h_1 + h_2}{\frac{h_2}{h_1} + 1}$$

Using this fact to calculate derivatives and inserting known values gives:

$$\sigma_{h_1}{}^2 = (5 \times 10^{-9})\left(\frac{1}{1 + \frac{h_2}{h_1}}\right)^2 + (3 \times 10^{-7})\left(-\left(\frac{h_1 + h_2}{\left(1 + \frac{h_2}{h_1}\right)^2}\right)\right)^2$$

Inserting values for h_1 and h_2 and calculating leads to:

$$\sigma_{h_1}{}^2 = (5 \times 10^{-9})(0.27693)^2 + (3 \times 10^{-7})(0.22385)^2 = 2 \times 10^{-8} m^2$$

Repeating the same process for the uncertainty in h_2 leads to:

$$\sigma_{h_2}{}^2 = \sigma_{h_1}{}^2\left(\frac{\partial h_2}{\partial h_1}\right)^2 + \sigma_{h_1+h_2}{}^2\left(\frac{\partial h_2}{\partial(h_1+h_2)}\right)^2 = \sigma_{h_1}{}^2(-1)^2 + \sigma_{h_1+h_2}{}^2(1)^2$$

$$\sigma_{h_2}{}^2 = (2 \times 10^{-8}) + (5 \times 10^{-9}) = 2 \times 10^{-8} + 3 \times 10^{-8} = 5 \times 10^{-8} m^2$$

Uncertainty in the difference (h_1-h_2) is then:

$$\sigma_{h_1-h_2}{}^2 = \sigma_{h_1}{}^2 + \sigma_{h_2}{}^2 = 3 \times 10^{-8} m^2$$

Calculating partial derivatives:

$$Let\ \omega = \left(\frac{T_1{}^2 + T_2{}^2}{2(h_1 + h_2)} + \frac{T_1{}^2 - T_2{}^2}{2(h_1 - h_2)}\right)^{-1} = 0.2494\ ms^{-2}$$

$$\frac{\partial g}{\partial T_1} = -4\pi^2\omega^2\left(\frac{T_1}{(h_1 + h_2)} + \frac{T_1}{(h_1 - h_2)}\right) = 6.7920$$

$$\frac{\partial g}{\partial T_2} = -4\pi^2\omega^2\left(\frac{T_2}{(h_1 + h_2)} - \frac{T_2}{(h_1 - h_2)}\right) = -17.7309$$

$$\frac{\partial g}{\partial h_1} = -4\pi^2\omega^2\left(-2\left(\frac{T_1{}^2 + T_2{}^2}{\left(2(h_1 + h_2)\right)^2}\right) - 2\left(\frac{T_1{}^2 - T_2{}^2}{\left(2(h_1 - h_2)\right)^2}\right)\right) = 12.1966$$

$$\frac{\partial g}{\partial h_2} = -4\pi^2\omega^2\left(-2\left(\frac{T_1{}^2 + T_2{}^2}{\left(2(h_1 + h_2)\right)^2}\right) + 2\left(\frac{T_1{}^2 - T_2{}^2}{\left(2(h_1 - h_2)\right)^2}\right)\right) = 12.1755$$

Inserting all values into the equation for the uncertainty in g gives:

$$\sigma_g{}^2 = (1.6 \times 10^{-9})(6.7920)^2 + (4 \times 10^{-10})(-17.7309)^2 + (2 \times 10^{-8})(12.1966)^2 + (3 \times 10^{-8})(12.1755)^2$$

$$\sigma_g{}^2 = 8 \times 10^{-6} m^2 s^{-4}$$

And so uncertainty in gravity is:

$$\sigma_g = 3 \times 10^{-3} ms^{-2}$$

Section E – Derivation of h_1 , h_2 and the difference ($h_1 - h_2$)

Utilising the ratio relationship between the heights found via the scales:

$$\frac{h_2}{h_1} = 2.611 \quad \rightarrow \quad h_2 = 2.611 h_1$$

Then:

$$h_1 + h_2 = h_1 + 2.6110 h_1 = 80.832cm \pm 0.007cm$$

$$h_1 = \frac{80.8315cm}{2.6110 + 1} = 22.3848cm \pm 0.01cm$$

Substitution leads to a value for h_2:

$$h_2 + h_1 = 80.8315cm \ \rightarrow \ h_2 = 80.8315cm - 22.3848cm = 58.4467cm \pm 0.02cm$$

And so a value for the difference ($h_1 - h_2$) is then:

$$h_1 - h_2 = 22.3848cm - 58.4467cm = -36.0619cm \pm 0.02cm$$

Calculations of uncertainties are included in appendix section D.